ARithmetische Geometrie OberSeminar
Arthur's Endoscopic Classification and Level One Cusp Forms
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In this Argos we want to learn about Arthur's conjectures and theorems on the classification of automorphic representations, by going through some concrete calculations.
Specifically, Chenevier and Renard in [CR] calculate the number of level one, polarized, algebraic regular, cuspidal automorphic representations of GL_n for small n using Arthur's classification: Let $n \geq 1$ be an integer and let π be a cuspidal automorphic representation of GL_n over $\mathbb{Q}$, such that
(a) (polarization) $\pi^\vee \cong \pi \otimes |\cdot|^w$ for some $w \in \mathbb{Z}$,
(b) (conductor 1) π_p is unramified for each prime p,
(c) π_∞ is algebraic and regular.
Then we want to calculate the number of such representations for $n \leq 6$ as a function of the *weights* of π_∞, which are integers $k_1 > k_2 > \cdots > k_n$ obtained from π_∞ via its infinitesimal character. The main goal is to understand the proof of the following theorem.

Theorem.** *Let $n \leq 6$. Then there is an explicit computable formula for the number $N(k_1, \ldots, k_n)$ of cuspidal automorphic representations π of GL_n satisfying $(a), (b)$ and (c) above and of weights $k_1 > k_2 > \cdots > k_n$.*

Following [CR], the two stars ** indicate that the result is conditional on the assumptions made in [A], especially on inner forms.
The above theorem has been proved by Chenevier and Renard in [CR] as a consequence of Arthur's multiplicity formula [A]. They also handle some bigger n, and give some interesting applications.

Let us give a very brief introduction to a greatly simplified version of Arthur's multiplicity formula. Consider a semisimple $\mathbb{Z}$-group G such that $G(\mathbb{R})$ is compact. Let $\mathcal{L}_\mathbb{Z}$ denote the (conjectural) Langlands group of $\mathbb{Z}$ and let

$$\psi : \mathcal{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$$

be a global Arthur parameter such that ψ_∞ is an Adams-Johnson parameter. Denote by π_ψ the irreducible admissible representation of $G(\mathbb{A})$ which is $G(\widehat{\mathbb{Z}})$-spherical and with the Satake parameters and infinitesimal character determined by ψ according to Arthur's recipe. This uniquely determines π_ψ as $G(\mathbb{R})$ is compact and connected.
Conjecturally any automorphic representation of G which is $G(\widehat{\mathbb{Z}})$-spherical has the form π_ψ for some ψ as above.
Denote also by $e(\psi)$ the (finite) number of $\widehat{G}$-conjugacy classes of global Arthur parameters ψ' as above such that $\pi_{\psi'} \simeq \pi_\psi$ (for most ψ we have $e(\psi) = 1$).

Then the multiplicity $m(\pi_\psi)$ of π_ψ in $\mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is conjectured to be

$$m(\pi_\psi) = \begin{cases} e(\psi) & \text{if } \rho^\vee_{|\mathrm{C}_\psi} = \varepsilon_\psi, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Here the group C_ψ is by definition the centralizer of $\mathrm{Im}\,\psi$ in $\widehat{G}$, which is a finite group. The character ε_ψ is defined by Arthur (see [A]). To define the character $\rho^\vee$, let $\varphi_{\psi_\infty} : \mathrm{W}_\mathbb{R} \longrightarrow \widehat{G}$ be the Langlands parameter associated by Arthur to ψ_∞. First, the centralizer in $\widehat{G}$ of $\varphi_{\psi_\infty}(\mathrm{W}_\mathbb{C})$ is a maximal torus $\widehat{T}$ of $\widehat{G}$, so that $\varphi_{\psi_\infty}(z) = z^\lambda \overline{z}^{\lambda'}$ for some $\lambda \in \frac{1}{2}\mathrm{X}_*(\widehat{T})$ and all $z \in \mathrm{W}_\mathbb{C}$, and λ is dominant with respect to a unique Borel subgroup $\widehat{B}$ of $\widehat{G}$ containing $\widehat{T}$. Let $\rho^\vee$ denote the half-sum of the positive roots of $(\widehat{G}, \widehat{B}, \widehat{T})$. As G is semisimple over $\mathbb{Z}$ and $G(\mathbb{R})$ is compact, this is actually a character of $\widehat{T}$. Note that by construction, we have $\mathrm{C}_\psi \subset \widehat{T}$ and so formula (1) makes sense.

Using Arthur's multiplicity formula and various cases of Langlands functoriality we can then (in the final three talks) determine the conjectural numbers of automorphic representations in question. Here is a sample result that one obtains (see [CR, Corollary** 1.11]):

Corollary.** *(1) $N(k_1, \dots, k_5, 0)$ vanishes for $k_1 < 23$.*

(2) There are exactly 7 tuples $(k_1, \dots, k_5)$ with $k_1 = 23$ such that $N(k_1, \dots, k_5, 0)$ is non-zero:

$$(23, 18, 14, 9, 5, 0), (23, 19, 13, 10, 4, 0), (23, 19, 15, 8, 4, 0), (23, 20, 14, 9, 3, 0),$$
$$(23, 20, 16, 7, 3, 0), (23, 21, 13, 10, 2, 0), (23, 21, 17, 6, 2, 0)$$

and for all of them $N(k_1, \dots, k_5, 0) = 1$.

Talks

1) The degenerate Weyl character formula

After recalling the Weyl character formula explain the degenerate Weyl character formula following [CR, Section 2.2].

2) Classical semisimple groups over $\mathbb{Z}$ and their automorphic forms

Following [CR, 3.1-3.6] introduce the classical groups, in particular the Chevalley groups $\mathrm{SO}_{p,q}$ and the definite groups SO_n. Prove that the class number $h(G)$ of a Chevalley group G is one [CR, Prop. 3.5]. Introduce the automorphic forms of level one of classical semisimple groups and describe the special case of definite groups [CR, Prop. 3.6].

3) Classification of the discrete series representations of real groups

Explain Langlands' classification of the discrete series representations of real groups, i.e. define discrete series L-parameters and their associated L-packets (see [K, Section 7]).

4) Endoscopy
Define endoscopic data as in [KS, Chapter 2] (see also [A, Section 3.2]). Explain endoscopy for $\mathrm{SL}_2(\mathbb{R})$ ([L, Section 4]).

5) Adams–Johnson packets
Following [CR, Appendix A] introduce Adams-Johnson parameters and the associated packets as well as their parametrization.

6) Arthur's symplectic orthogonal alternative and global parameters
After recalling the Langlands parametrization of $\Pi_{disc}(G)$ describe Arthur's symplectic orthogonal alternative [CR, Section 3.8]. Prove the results of [CR, Section 3.11] and introduce global Arthur parameters, both in terms of the conjectural Langlands group, and the unconditional workaround for classical groups.

7) Global Arthur packets and the multiplicity formula
Describe the packet $\Pi(\psi)$ associated with a global Arthur parameter ψ ([CR, Section 3.21]). Explain the multiplicity formula given in [CR, Conjecture 3.30].

8) Symplectic forms of GL_4
Justify the formula for the number $S(v,w)$ of level 1, symplectic, polarized, algebraic regular, cuspidal automorphic representations of GL_4 of Hodge weights v,w. Moreover explain how to compare automorphic representations of groups that are related via a central isogeny [CR, Prop. 4.7].

9) Symmetric square and tensor product functoriality
Relate the dimensions of spaces of orthogonal forms and symplectic forms via Langlands functoriality by proving [CR, Theorem 1.15].

10) Level one forms of GL_6
Prove [CR, Theorem 1.5] in the case $n = 6$, see [CR, Chapter 5].

References

[A] Arthur, James, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, 61:xviii+590, 2013

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[KS] Kottwitz, Robert and Shelstad Diana, *Foundations of twisted endoscopy* Astérisque, 255:vi + 190, 1999.

[L] Labesse, Jean-Pierre, *Introduction to endoscopy*, *In* Representation theory of real reductive Lie groups, 472:175–213, Amer. Math. Soc., Providence, RI, 2008.