The transport Oka-Grauert principle for simple surfaces

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Overview 1/2: Range characterisations in inverse problems

Inverse problems are typically posed in terms of a *forward operator*

$$\mathcal{F}\colon \mathcal{X} \to \mathcal{Y}.$$

Often $\mathcal{F}^{-1}$ is not available, so we ask for injectivity, stability, ...

... & the range:

Problem: Characterise/understand the range $\mathcal{F}(\mathcal{X}) \subset \mathcal{Y}$.

Examples:

1. Helgason-Ludwig (1964): $\mathcal{F}$ = linear X-ray transform on $\mathbb{R}^n$ // range is characterised by moment conditions;
2. Pestov-Uhlmann (2004): $\mathcal{F}$ = linear X-ray transform on simple surface // range is parametrised by boundary operator;
3. Sharafutdinov (2011): $\mathcal{F}$ arising from Calderón problem on disk // elements of the range are related by conjugation;
4. Burago-Ivanov (2014): $\mathcal{F}$ = boundary distance map for Finsler metrics on n-ball // range is open under suitable perturbations;
5. **This talk:** $\mathcal{F}$ = non-Abelian X-ray transform on simple surface // nonlinear version of Pestov-Uhlmann result.

Overview 2/2: Connections to complex geometry

Common theme for some of these characterisations in 2D: Based on hard **transitivity theorem** with **complex geometric interpretation**.

	Transitivity theorem	Complex geometry
Calderón problem on the disk	any g is conformally flat	Riemann mapping theorem
Linear X-ray on simple surface	$\exists$ "scalar holomorphic integrating factors"	$H^{0,1}_{\bar\partial}(Z) = 0$
Non-Abelian X-ray on simple surface	$\exists$ "matrix holomorphic integrating factors"	Transport Oka-Grauert principle: $\mathfrak{M}(Z) = 0$
	$\iff$ transitivity of a certain group action	We introduce a novel transport twistor space Z

Structure of talk: ●●● → ●● → ●●●●●

Let (M, g) be a compact Riemannian surface with boundary ∂M. Assume that ∂M is strictly convex and that M is *non-trapping* ($\Rightarrow M \approx$ disk).

On $SM = \{(x, v) \in TM : g(v, v) = 1\}$ consider the **transport equation**

$$(X + \mathbb{A})R = 0 \text{ on } SM, \tag{TE}$$

with $X =$ *geodesic vector field* and $\mathbb{A} \in C^\infty(SM, \mathbb{C}^{n\times n})$ an *attenuation.*

Note: $R \in C^\infty(SM, \mathbb{C}^{n\times n})$ solves (TE), iff $\forall$ geodesics $\gamma\colon [0, \tau] \to M$,

$$\frac{d}{dt} R(\gamma(t), \dot\gamma(t)) + \mathbb{A}R(\gamma(t), \dot\gamma(t)) = 0. \tag{TE'}$$

Let $\partial_\pm SM = \{(x, v) \in SM : x \in \partial M, \pm g(v, \nu(x)) \geq 0\} =$ influx /outflux.

Definition

Let $R =$ unique solution of (TE) with $R|_{\partial_- SM} = \mathrm{Id}$, define:

$$C_\mathbb{A} = R|_{\partial_+ SM} \in C^\infty(\partial_+ SM, GL(n, \mathbb{C})) \quad \rightsquigarrow \quad \text{scattering data of } \mathbb{A};$$

$$\mathbb{A} \mapsto C_\mathbb{A} \quad \rightsquigarrow \quad \text{non-Abelian X-ray trafo.}$$

Examples:

- Scalar case ($n = 1$): $C_{\mathbb{A}} = \exp(I\mathbb{A})$, where $I =$ linear X-ray transform;
- Connections: If $\mathbb{A}(x, v) = A_x(v)$ for 1-form $A \in \Omega^1(M)$, then

$$C_A = \text{parallel transport of connection } d + A \text{ on } M \times \mathbb{C}^n;$$

- Polarimetric Neutron Tomography: If $\mathbb{A}(x, v) = \Phi(x) \in \mathfrak{so}(3)$, then

$$C_\Phi = \text{ spin rotation in } SO(3) \text{ of neutrons after traversing } \vec{B} \text{ field.}$$

Theorem (Paternain-Salo-Uhlmann 2012 & 2020)

Let (M, g) be simple (i.e. ∂M strictly convex, non-trapping & no conjugate points). Suppose $\mathbb{A}(x, v) = A_x(v) + \Phi(x)$ and $\mathbb{B} = B_x(v) + \Psi(x)$ are s.th.

$$C_{\mathbb{A}} = C_{\mathbb{B}}.$$

Then there exists a gauge $\varphi \in C^\infty(M, GL(n, \mathbb{C}))$ with $\varphi = \mathrm{Id}$ on ∂M and

$$\Phi = \varphi^{-1}\Psi\varphi, \quad A = \varphi^{-1}d\varphi + \varphi^{-1}B\varphi.$$

Theorem (B.-Paternain)

Let (M, g) be a simple surface and $q \in C^\infty(\partial_+ SM, U(n))$, then TFAE:

1. $q = C_{\mathbb{A}}$ *for some* $\mathfrak{u}(n)$*-valued* $\mathbb{A} = \Phi + A$*;*
2. q *lies in the range of a* **boundary operator**

$$P\colon C^\infty(\partial_+ SM, \mathbb{C}^{n\times n}) \supset D(P) \to C^\infty(\partial_+ SM, U(n)).$$

- Nonlinear version of Pestov-Uhlmann (2004);
- P defined in terms of Birkhoff factorisation; *morally* its domain is

$$D(P) \approx \begin{matrix}\text{Hermitian metrics} \\ \text{on } \partial_+ SM \times \mathbb{C}^n\end{matrix} \approx \begin{matrix}\text{Radiative/dispersive} \\ \text{degrees of freedom (DOF);}\end{matrix}$$

- Analogy with Ward correspondence by Mason (2006):

$$\begin{matrix}\text{Solutions to} \\ \text{ASDYM}\end{matrix} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix}\text{Solitonic} \\ \text{DOF}\end{matrix} \right\} \times \left\{ \begin{matrix}\text{Radiative/dispersive} \\ \text{DOF}\end{matrix} \right\}$$

- TOG principle: $\nexists$ nontrivial holomorphic vector bundles on Z.

Matrix holomorphic integrating factors

Any $F \in C^\infty(SM, \mathbb{C}^{n\times n})$ has *vertical Fourier decomposition* $F = \sum_{k\in\mathbb{Z}} F_k$. We call F *fibrewise holomorphic*, iff $F_k = 0$ for $k < 0$. Define

$$\mathbb{G} = \{F \in C^\infty(SM, GL(n,\mathbb{C})) : F, F^{-1} \text{ are fibrewise holomorphic}\}.$$

Definition

A holomorphic integrating factor for $\mathbb{A}$ is a solution $F \in \mathbb{G}$ to $(X + \mathbb{A})F = 0$.

- Why: Gauge respecting Fourier support & P yields only $\mathbb{A}'s$ with HIF;
- existence for $n = 1$ on simple surfaces due to Salo-Uhlmann (2011);
- existence for $n \geq 2$ was largely open (*weak* solutions in Euclidean setting due to Novikov (2002) and Eskin-Ralston (2004));
- necessary condition (satisfied for $\mathbb{A} = A + \Phi$): $\mathbb{A}$ lies in the set

$$\mho = \{\mathbb{A} \in C^\infty(SM, \mathbb{C}^{n\times n}) : \mathbb{A}_k = 0 \text{ for } k < -1\}.$$

Theorem (B.-Paternain)

Let (M, g) be simple. Then any $\mathbb{A} \in \mho$ has holomorphic integrating factors.

Recall:

$$\begin{aligned} \mathbb{G} &= \{F \in C^\infty(SM, GL(n,\mathbb{C})) : F, F^{-1} \text{ are fibrewise holomorphic}\} \\ \mho &= \{\mathbb{A} \in C^\infty(SM, \mathbb{C}^{n\times n}) : \mathbb{A}_k = 0 \text{ for } k < -1\} \end{aligned}$$

Proof of theorem.

- $\mathbb{G}$ is a group that acts on $\mho$ via $(\mathbb{A}, F) \mapsto F^{-1}(X + \mathbb{A})F$ such that

$$\begin{aligned} \mathbb{A} \text{ admits HIF} \quad &\iff \quad \mathbb{A} \text{ lies on same } \mathbb{G}\text{-orbit as } 0, \\ \text{Theorem} \quad &\iff \quad \mathbb{G} \text{ acts transitively on } \mho; \end{aligned}$$

- Key step: The derivative of $F \mapsto \mathbb{A}.F$ at Id, given by

$$T_{\mathrm{Id}}\mathbb{G} \to \mho, \quad H \mapsto XH + [\mathbb{A}, H]$$

  is **onto** and has a **tame right inverse**. This uses results on the attenuated X-ray transform $I_\mathbb{A}$ & microlocal analysis of $I_\mathbb{A}^* I_\mathbb{A}$;
- Nash-Moser IFT $\Longrightarrow$ $\mathbb{G}$-orbits are open $\Longrightarrow$ action is transitive. □

Note: Original motivation for matrix HIF was to prove injectivity of $I_\mathbb{A}$ (up to gauge); we go the other way!

We set up a correspondence for **any** orientable Riemannian surface:

$$\begin{aligned} (M,g) &\rightsquigarrow \text{(degenerated) complex surface } Z; \\ \mathbb{A} &\rightsquigarrow \text{holomorphic vector bundle over } Z. \end{aligned}$$

Idea: Fill in the disks in SM and extend X to Cauchy-Riemann operator.

The transport twistor space

The 4-manifold $Z = \{(x,v) \in TM : g(v,v) \leq 1\}$ supports a natural complex distribution $D \subset T_{\mathbb{C}}Z$ of rank 2 that is involutive and satisfies

$$D \cap \bar{D} = \begin{cases} \operatorname{span}_{\mathbb{C}} X & \text{on } SM \\ 0 & \text{on } Z \backslash SM. \end{cases}$$

In particular, Z^{int} is a complex surface with $T^{0,1}Z^{\text{int}} = D$.

- Construction extends to other flows on SM (e.g. magnetic flows);
- Z is branched double cover of *classical twistor space* from Dubois-Violette (1983) and O'Brian-Rawnsley (1985).

Example: Suppose $M \subset \mathbb{C}$ with Euclidean metric, then

$$SM = \{(z, \mu) \in \mathbb{C}^2 : z \in M, |\mu| = 1\}.$$

Write $z = x + iy$ and $\mu = \cos\theta + i\sin\theta$, then

$$X = \cos\theta \cdot \partial_x + \sin\theta \cdot \partial_y = \mu\partial_z + \bar{\mu}\partial_{\bar{z}} = \bar{\mu}\left(\mu^2\partial_z + \partial_{\bar{z}}\right).$$

Definition

On $Z = \{(z, \mu) \in \mathbb{C}^2 : z \in M, |\mu| \leq 1\}$ we define $D \subset T_{\mathbb{C}}Z$ by

$$D = \operatorname{span}_{\mathbb{C}}\left\{\mu^2\partial_z + \partial_{\bar{z}}, \partial_{\bar{\mu}}\right\}.$$

Say $f \in C^\infty(U)$ is *holomorphic* iff $(\mu^2\partial_z + \partial_{\bar{z}})f = \partial_{\bar{\mu}}f = 0$ on $U \subset Z$ open.

- $[D, D] = 0$ and $D \cap \bar{D} = \operatorname{span}_{\mathbb{C}} X$ for $|\mu| = 1$ are immediate;
- to incorporate different geometries/flows, replace X with $F = X + \lambda V$. If $\mu^2\lambda(z, \mu)$ is μ-holomorphic, then D is well defined by

$$D = \operatorname{span}_{\mathbb{C}}\left\{\mu^2\partial_z + \partial_{\bar{z}} + i\mu^2\lambda\partial_\mu, \partial_{\bar{\mu}}\right\};$$

- description in isothermal coordinates, but D is defined invariantly.

Notions of complex geometry (e.g. $\bar{\partial}$-complex, Dolbeaut cohomology, holomorphic vector bundles) are defined on Z *smooth up to the boundary.*

Let $\oplus_{k\geq k_0}\Omega_k = \{u \in C^\infty(SM) : u_k = 0 \text{ for } k < k_0\}$ and note that

$$X\colon \oplus_{k\geq 0}\,\Omega_k \to \oplus_{k\geq -1}\Omega_k.$$

Theorem (Correspondence principle A)

The twistor space of **any** *Riemannian surface* (M,g) *satisfies*

$$H^{0,p}_{\bar{\partial}}(Z) \cong \begin{cases} \{u \in \oplus_{k\geq 0}\Omega_k : Xu = 0\} & p = 0, \\ \oplus_{k\geq -1}\Omega_k / X(\oplus_{k\geq -1}\Omega_k) & p = 1, \\ 0 & p \geq 2. \end{cases}$$

- $p = 0$: fibrewise holomorphic first integrals;
- $p = 1$: solvability of $Xu = f$ for fibrewise holomorphic u;
- Salo-Uhlmann (2011) $\leftrightarrow$ if (M,g) is simple, then $H^{0,1}_{\bar{\partial}}(Z) = 0$;
- trapping produces non-trivial elements in degree $p = 1$;

Assume for simplicity that $M \approx$ disk, such that all vector bundles are topologically trivial.

Theorem (Correspondence principle B)

1. *For any attenuation* $\mathbb{A} \in \mho$ *there exists a holomorphic vector bundle* $E_{\mathbb{A}} \to Z$ *such that* $H^p(Z, E_{\mathbb{A}})$ *is given in terms of* $(X + \mathbb{A})$*;*
2. *any holomorphic vector bundle is isomorphic to* $E_{\mathbb{A}}$ *for some* $\mathbb{A} \in \mho$*;*
3. *the moduli space of holomorphic vector bundles equals*

$$\mathfrak{M}(Z) \equiv \left\{ \begin{array}{l} \textit{holomorphic rank } n \textit{ vector bundles} \\ \textit{on } Z, \textit{ up to isomorphism} \end{array} \right\} \cong \mho/\mathbb{G}.$$

Theorem (TOG principle, B.-Paternain)

If Z *is the twistor space of a simple surface* (M, g)*, then* $\mathfrak{M}(Z) = 0$.

- Oka-Grauert principle: On a Stein manifold, the classification of holomorphic vector bundles equals that of topological vector bundles.

Cohomology computations & TOG-principle suggest the following slogan:

> *The twistor space of a simple surface behaves like a (contractible) Stein surface.*

Question: In the simple case, is Z^{int} actually a Stein surface?

Examples:

- If $M = \mathbb{R}^2$, then there is explicit blow down map $\beta\colon Z \to \mathbb{C}^2$, s.th.

$$Z^{\mathrm{int}} \cong \beta(Z^{\mathrm{int}}) = \text{polydisk in } \mathbb{C}^2 \quad \Longrightarrow \quad Z^{\mathrm{int}} \text{ is Stein;}$$

- if Z is the twistor space of a constant magnetic field on $\mathbb{R}^2$, then

$$Z^{\mathrm{int}} \backslash 0 \cong \mathbb{C}^2 \backslash \{\bar{w}_1 = w_2\} \quad \Longrightarrow \quad Z^{\mathrm{int}} \text{ is } \textit{not} \text{ Stein.}$$

Thank you for your attention!

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Appendix A: Birkhoff factorisation & definition of P

Let Her_+^n = Hermitian positive definite $n \times n$ matrices.

Theorem (Symmetric Birkhoff factorisation)

For any $H \in C^\infty(SM, \mathrm{Her}_+^n)$ there exists $F \in \mathbb{G}$ such that $H = F^* F$.

Let $\alpha \colon \partial SM \to \partial SM$ be the scattering relation.

How to generate elements in the range of $C^\infty(M, \mathfrak{u}(n)) \ni \Phi \mapsto C_\Phi$:

1. Start with $w \in D(P) := C_\alpha^\infty(SM, \mathrm{Her}_+^n)$;
2. extend to first integral $w^\sharp \in C^\infty(SM, \mathrm{Her}_+^n)$;
3. factor as $w^\sharp = F^* F$ (unique after requiring $F_0 = \mathrm{Id}$);
4. let $\Phi = -(XF)F^{-1} \in C^\infty(M, \mathfrak{u}(n))$, then

$$C_\Phi = Pw := F|_{\partial SM} \circ (F^{-1}|_{\partial SM} \circ \alpha) \quad \text{on } \partial_+ SM.$$

Recall: The Cauchy Riemann equations on $Z(\mathbb{R}^2) \equiv \mathbb{C}_z \times \mathbb{D}_\mu$ are

$$(\mu^2 \partial_z + \partial_{\bar{z}}) f = 0 \quad \text{and} \quad \partial_{\bar{\mu}} f = 0.$$

The blow down map

The following map is holomorphic:

$$\beta \colon Z \to \mathbb{C}^2, \quad \beta(z, \mu) = (z - \mu^2 \bar{z}, \mu)$$

It has a partial inverse given by

$$\beta^{-1}(w, \mu) = \left(\frac{w}{1 + |\mu|^2} + \frac{2\,\mathrm{Re}(\bar{\mu} w)}{1 - |\mu|^4}, \mu \right), \quad (w, \mu) \in \beta(Z) \backslash \{|\mu| = 1\}.$$

- Original approach of Eskin-Ralston (2004) to obtain HIF: Use β to desingularise Z and apply the classical Oka-Grauert principle on $\beta(Z)$.