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Étude mathématique de fluides en interaction avec un champ magnétique

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For [God] looketh to the ends of the earth, and seeth under the whole heaven;
To make the weight for the winds; and he weigheth the waters by measure.
When he made a decree for the rain, and a way for the lightning of the thunder:
Then did he see [Wisdom], and declare it; he prepared it, yea, and searched it out.
And unto man he said, Behold, the fear of the Lord, that is wisdom; and to depart from evil is understanding.

Job 28.24-28

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Remerciements

Les mathématiques, ça ne s'assène pas : ça se raconte.
Denis Choimet[1]

Les études mathématiques sont bien souvent perçues comme étant le sommet de l'objectivité, où règne en seul maître une raison pure et impersonnelle. Après tout, la vérité mathématique ne doit-elle pas être au delà des opinions divergentes et des querelles des hommes ? Cependant, si la rigueur et la précaution font bien la fierté de la discipline, elle n'a rien d'impersonnel. Chaque découverte mathématique est une étape sur le chemin de joie, de peines, d'acharnement, ou perplexité qu'empruntent les mathématiciens. Et ils sont rarement solitaires sur ce chemin.

Ainsi, les efforts de nombreuses personnes se cachent derrière ces pages. Rien de ce qui suit ne devrait être pris indépendamment de ceux qui m'ont aidé et amené jusqu'ici. Ce travail est tout autant leur travail, qui reçoit ici une fraction de la reconnaissance que je leur dois.

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[1] Entendu dans l'un de ses cours.

La bonne entente existant parmi les doctorants a été une précieuse motivation : sans pouvoir discuter et partager les choses de la vie (dont les mathématiques !), le travail aurait été bien solitaire. Merci à tous ceux qui ont partagé leurs repas avec moi et ont mis de la joie dans les bureaux ! Parmi eux se trouvent Gauthier, Simon, Valentin (flûtiste invétéré), Sébastien, Thibault, Marianne, Léa, Martin, Octave, Mélanie, Marina, Sébastien, Daniel et Jorge. Remerciements particuliers à ceux qui ont eu (ont du avoir ?) la patience de partager un bureau avec moi : Caterina, Tanessi, Anatole, Benjamin, Uran, Julien, Adrien, Annette et Luca.

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En contemplant le chemin parcouru, je vois à quel point le rôle des enseignants est grand dans la formation d'une personne. Plus encore quand ceux-ci ne se content pas d'être préoccupés par la transmission des connaissances (ce qu'il font déjà merveilleusement bien), mais qu'ils ont de plus à coeur le bien-être et le développement de leurs étudiants en tant que personnes. Merci en particulier à Jean-Luc Joly, Patrick Séguin, Denis Choimet Jacques Renault, Romain Presle, Karine Beauchard, Benjamin Boutin et Jérémy Leborgne. Je tiens à exprimer une gratitude toute particulière à Denis Choimet qui a été déterminant dans mon parcours en m'apprenant patiemment que *les mathématiques, ça ne s'assène pas : ça se raconte*. Et aussi pour avoir su communiquer sa passion pour l'analyse (comme quoi *les égalités sont bien sympathiques, mais les inégalités, ce n'est pas trop mal non plus*).

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List of Articles

This is the list of articles that were written during the PhD period. The dissertation is largely based on these, although it also contains some original material. The articles are sorted in chronological order of writing. In the bibliography, they are referred to as, respectively, [28], [29], [30], [25] and [26].

- D. Cobb and F. Fanelli : *Rigorous derivation and well-posedness of a quasi-homogeneous ideal MHD system.* Nonlinear Anal. Real World Appl. 60 (2021), Paper No. 103284, 36 pp.
- D. Cobb and F. Fanelli : *Elsässer formulation of the ideal MHD and improved lifespan in two space dimensions.* arXiv :2009.11230 (2020) (submitted).
- D. Cobb and F. Fanelli : *Symmetry breaking in ideal magnetohydrodynamics : the role of the velocity.* J. Elliptic Parabol. Equ. 7 (2021), no. 2, 273–295.
- D. Cobb : *Bounded Solutions in Incompressible Hydrodynamics.* arXiv :2105.03257, 2021 (submitted).
- D. Cobb : *Remarks on Chemin's Space of Homogeneous Distributions.* (in preparation)

Our article [27], which was published during the PhD, is not covered in this dissertation, as it has already been presented in a master's thesis. The last article [26] is still in preparation.

Notation and Conventions

In this page, we introduce the symbols that are the most common throughout the dissertation. Other less used notation will be defined as we go on. In order to help the reader cruise in the ocean of confused equations below, an index is provided at the end of the manuscript.

Constants and inequalities : in all that follows, C is a generic constant that may change value from one line to another. When needed, we will specify the useful dependencies of the constant by the notation $C(\,.\,)$. If a and b are two nonnegative quantities, we note $a \lesssim b$ for $A \leq Cb$ and $a \approx b$ for

$$\frac{1}{C} a \leq b \leq Ca.$$

Banach spaces and norms : if X is a Banach space, we note $\|\,.\,\|_X$ the associated norm. Two Banach spaces X and Y are said to be isomorphic if there is a linear isomorphism $\phi : X \to Y$ that is also a quasi-isometry : $\|\,.\,\|_X \approx \|\,.\,\|_Y$. If $F \subset X$ is a closed subspace of a Banach space X, then X/F is a Banach space when equipped with the norm

$$\forall x \in X/F, \qquad \|x\|_{X/F} = \inf_{f \in F} \|x - f\|_X.$$

Functions and classes of functions : as usual with PDEs, notation is a jumbled mess : "functions" are in fact defined up to a measure zero set, variables are taken almost everywhere in σ-algebras which may differ from one line to the next, there is a systematic identification between functions and bounded linear functionals, *etc.* As much as possible (and that is very little at all), we strive to be consistent with our choices of notation.

The symbols f or $f(x)$ will note the class of functions equal to $f(x)$ for almost every x. Identification almost everywhere is, unless otherwise mentioned, relative to the Borel σ-algebra of $\mathbb{R}^d$.

Function spaces : unless otherwise mentioned, all functions spaces are set on $\mathbb{R}^d$. In order to simplify notations, we will omit the reference to $\mathbb{R}^d$ when writing such a space. For example, we note $L^p(\mathbb{R}^d) = L^p$.

Sometimes, we will consider L^p functions defined on an interval $[0, T]$, where $T \geq 0$, and with values in a Banach space X. We note $L^p([0, T]; X) = L^p_T(X)$ the space of such functions. We will use the shorthand $L^p(X) = L^p(\mathbb{R}_+; X)$.

Sequence spaces : For $p \in [1, +\infty]$, we note $\ell^p = \ell^p(\mathbb{N})$ the classical sequence spaces. Occasionally, we will consider sequences defined on $\mathbb{Z}$ or $\{-1, 0, 1, ...\}$, but it will always be clear from the context. The space $c_0 = c_0(\mathbb{N})$ is the set of sequences that converge to zero.

Distribution spaces : we use the Schwartz notation for distribution spaces : $\mathcal{D}(\Omega)$ is the space of C^∞ and compactly supported functions on an open subset $\Omega \subset \mathbb{R}^d$, $\mathcal{S}$ is the class of Schwartz functions, $\mathcal{D}'(\Omega)$ is the space of distributions on Ω and $\mathcal{S}'$ the space of tempered distributions.

Duality : if X is a Frechet space, we will note X' its topological dual and $\langle\,.\,,\,.\,\rangle_{X'\times X}$ the duality bracket. In the case of function spaces, we will always identify X' with the space of functions it is isomorphic to. Hence, we do note hesitate to write the equality $(L^1)' = L^\infty$.

Derivatives : unless otherwise mentioned, (pseudo-)differential operators (for example ∇, Δ, curl, div and $(\Delta)^{-1}$) refer exclusively to the space variable. If $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a vector field, ∇f refers to the transpose of the differential matrix, in other words $[\nabla f]_{ij} = \partial_i f_j$.

Sums : most, if not all, the equations we study come directly from physical systems, and feature many sums of the form $\sum_k f_k \partial_k g$. When we believe it provides better legibility, we use the following summation convention : there is an implicit sum on any repeated index. For example, the previous sum will be noted simply $f_k \partial_k g$, the sum ranging on $k = 1, ..., d$. Note that, as we only work in the *Euclidean* space $\mathbb{R}^d$, there is no point in distinguishing between covariant and contravariant quantities, so all indices will be noted as subscripts.

Résumé en français

Le problème avec le futur, c'est qu'il n'arrête pas de devenir le présent.
Bill Watterson[2]

Avant-propos

Cette thèse de doctorat est l'aboutissement de trois années de recherche concernant les propriétés mathématiques de la magnétohydrodynamique (que nous abrégerons MHD). Comme son nom l'indique, la MHD traite des fluides en interaction avec un champ magnétique qu'ils génèrent grâce à leurs propriétés conductrices de courant électrique. Le domaine de la MHD est né au début des années $40'$ avec les travaux du physicien Hannes Alfvén, et est un des objets d'étude des mathématiques depuis les années $80'$.

Le lecteur l'aura compris, la MHD n'est certainement pas un terrain vierge dans le paysage scientifique. Malgré la diversité extraordinaire des applications, les propriétés intéressantes et l'abondance des questions ouvertes, de très nombreux problèmes en MHD ont déjà fait l'objet d'intenses recherches. Ceux que nous étudions dans cette thèse ne font pas exception. C'est pourquoi nous ne pouvons prétendre à l'exhaustivité tant dans nos explications que notre bibliographie, même si, très humblement, nous essayerons de donner un aperçu de la recherche actuelle sur les questions que nous aborderons.

L'un des facteurs qui a certainement contribué à l'abondance des publications est que la MHD a parfois été traitée comme une simple généralisation des équations d'Euler ou de Navier-Stokes. Il peut être extrêmement tentant de mener des recherches en MHD en se limitant seulement à étendre à ce "cas général" les résultats qui ont été démontrés dans le "cas particulier" des équations d'Euler ou de Navier-Stokes. Cette approche a comme corollaire malheureux d'inciter le mathématicien à négliger la physique très riche de la MHD. Celle-ci possède de nombreux éléments qui lui sont propres, tels que, entre autres, des quantités conservées, des solutions statiques non-triviales, des questions de stabilité délicates ou une théorie des solutions bien plus complexe que celle des équations d'Euler.

Bien que nous n'ayons pas la prétention de nous affranchir totalement de cette philosophie délétère de la MHD décrite ci-dessus, nous espérons toutefois que ces remarques pourront jeter un peu de lumière sur ce travail de thèse. Notre objectif sera de parvenir à une compréhension plus profonde de la MHD en mettant en avant les différentes structures qui lui sont spécifiques ainsi que les méthodes mathématiques "taillées sur mesure" aux problèmes que nous étudierons. Nous reviendrons parfois sur des résultats connus en donnant, le cas échéant, des précisions et généralisations, et nous présenterons parfois nos propres travaux sur la question.

Enfin, il nous faut signaler que ce résumé en français n'est que l'ombre de l'introduction en anglais, plus longue, qui couvre une présentation générale des différents aspects de la théorie, tant mathématiques que du point de vue de la physique. Ainsi, nous encourageons le lecteur anglophile à s'y référer de préférence.

[2]Traduit de l'anglais d'après Bill Watterson. *The problem with the future is that it keeps turning into the present.*

Position du problème

Le corps de ce travail de thèse concerne la question de l'existence et l'unicité des solutions au problème de Cauchy pour les équations de la MHD idéale. Il s'agit d'un système d'EDP qui s'écrivent

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla \pi = (b \cdot \nabla)b \\ \partial_t b + (u \cdot \nabla)b = (b \cdot \nabla)u \\ \operatorname{div}(u) = 0, \end{cases} \tag{1}$$

Nous étudierons ces équations dans tout l'espace euclidien $\mathbb{R}^d$. Ce choix, comme nous le verrons, est lourd de conséquences, tant sur les méthodes que nous emploierons que sur la nature précise de nos résultats. L'espace $\mathbb{R}^d$ n'est pas que le cadre propice aux techniques d'analyse de Fourier que nous mettrons en oeuvre (auquel cas se placer dans le tore $\mathbb{T}^d$ conviendrait tout autant). En réalité, le simple fait de travailler avec des fluides incompressibles d'extension spatiale infinie aura des conséquences profondes sur notre analyse.

Dans les équations (1), $u(t, x) \in \mathbb{R}^d$ est le champ des vitesses moyen des particules de fluide au point $x \in \mathbb{R}^d$ et à l'instant $t \in \mathbb{R}$, tandis que $b(t, x) \in \mathbb{R}^d$ est la valeur du champ magnétique. Le scalaire $\pi(t, x)$ est la pression MHD, une combinaison de la pression hydrodynamique habituelle en mécanique des fluides et la "pression magnétique"' liée à la force de Laplace (on pourra consulter (17) dans l'introduction en anglais).

La question de l'existence et l'unicité des solutions est un problème particulièrement difficile dans le domaine général des EDP. Un simple coup d'oeil aux équations sera suffisant pour s'en convaincre : Les équations pour u et b sont toutes les deux des équations de transport, ce qui signifie qu'elles possèderont la propriété de propagation de la régularité. En revanche, les second membres de ces équations sont moins réguliers que u et b, puisque impliquant des dérivées de ces mêmes quantités. Nous pourrions ainsi nous attendre à une dégénérescence instantanée des solutions, ce qui mettrait en péril la possibilité même de construire des solutions sur un court intervalle de temps.

Fort heureusement, il s'agit là d'une évaluation pessimiste de la réalité : une des propriétés fondamentales du système (1) permet d'éviter le désastre, à savoir la conservation de l'énergie. La somme de l'énergie cinétique totale et de l'énergie contenue dans le champ magnétique est une constante de la dynamique :

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \left\{ |u|^2 + |b|^2 \right\} \mathrm{d}x = 0.$$

Cette loi de conservation confère aux équations une structure dans laquelle nous pouvons travailler, celle des systèmes hyperboliques symétriques quasi-linéaires. L'existence et l'unicité des solutions est donc garantie dans les espaces de Sobolev sous critiques H^s, avec $s > 1 + d/2$.

Théorème. (Schmidt, 1987, [95]) *Soit $u_0, b_0 \in H^s$ deux fonctions de divergences nulles et $s > 1 + d/2$. Alors il existe un temps $T > 0$ et une unique solution $(u, b) \in C^0([0, T[; H^s)$ du système (1) qui soit associée à la donnée initiale (u_0, b_0).*

Malheureusement, cette méthode est limitée à des solutions évoluant dans des espaces de fonctions basés sur L^2, puisque elle repose sur des estimations d'énergie. Par ailleurs, il est connu que les systèmes hyperboliques symétriques généraux ne sont pas bien posés dans L^p pour d'autres valeurs de $p \neq 2$ (voir par exemple [15]). Cet obstacle est assez regrettable, puisque travailler avec des solutions dans des espaces basés sur L^p avec $p \neq 2$ présente de nombreux avantages, principalement concernant le niveau de régularité des solutions.

Vu la forme générale des équations (1) de la MHD idéale, nous recherchons des solutions qui soient lipschitziennes $W^{1,\infty}$. En termes d'injections entre espaces de fonctions (le lecteur pourra penser aux injections de Sobolev, même si nous travaillerons principalement avec des espaces de Besov), nous nous concentrerons sur des solutions dont l'exposant de régularité sera $s \geq 1 + d/p$. Par conséquent, travailler avec une exposant de Lebesgue le plus grand possible permettra d'obtenir des solutions ayant un exposant de régularité le plus faible possible. En vue d'expliquer quels avantages cela peut amener, penchons nous sur le cas extrême $p = +\infty$ et limitons nous aux équations d'Euler incompressibles dans le plan $d = 2$.

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla\Pi = 0 \\ \mathrm{div}(u) = 0. \end{cases} \tag{2}$$

Autrement dit, nous fixons le champ magnétique à $b \equiv 0$ dans (1). Parce que nous travaillons en deux dimensions d'espace, le tourbillon $\omega = \partial_1 u_2 - \partial_2 u_1$ est solution d'une équation de transport sans aucun second membre :

$$\partial_t \omega + u \cdot \nabla\omega = 0.$$

De façon ordinaire, pour des flots u de régularité $s = 1 + d/p$, c'est à dire pour des tourbillons de régularité $s - 1 = d/p$, la théorie des équations de transport[3] donne une inégalité exponentielle de la forme

$$\|\omega\|_{L^\infty_T(X^{s-1})} \leq \|\omega_0\|_{X^{s-1}} \exp\left(C \int_0^T \|\nabla u\|_{L^\infty} \,\mathrm{d}t \right),$$

où X^{s-1} est un espace de fonctions de régularité $s - 1 = d/p$. Cela contraste évidemment avec la simple conservation des normes de Lebesgue $\|\omega\|_{L^p} = \|\omega_0\|_{L^p}$ due à l'incompressibilité et au fait que L^p est un espace constitué de fonctions de régularité $s - 1 = 0$. En vue de se rapprocher le plus possible de cela, l'idée est de travailler avec $p = +\infty$, de sorte que $X^{s-1} = X^0$ soit un espace de fonctions de régularité $s - 1 = 0$, c'est à dire ressemblant le plus possible à l'espace de Lebesgue L^∞.

De nombreux théorèmes sur l'existence et l'unicité des solutions de (2) sont démontrés dans cet esprit, que ce soit ceux de Yudovich (*c.f.* le Chapitre 8 de [81]) ou de Serfati [97] qui dépendent de façon cruciale de la conservation de $\|\omega\|_{L^\infty}$, ou encore le résultat de Hmidi et Keraani [64] construit sur le même principe.

Le rôle des variables d'Elsässer

Nous revenons maintenant à la MHD idéale (1). Jusqu'ici, nous avons spéculé sur les avantages que nous aurions à avoir des solutions dans des espaces basés sur L^p avec $p \neq 2$, tout en précisant l'insuffisance de la théorie hyperbolique usuelle pour arriver à nos fins. Nous devons donc recourir à des propriétés plus profondes des équations, qui peuvent être dévoilées par un changement de variables très simple qui porte le nom du physicien germano-américain W.M. Elsässer qui l'a introduit en 1950. Posons

$$\alpha = u + b \qquad \text{and} \qquad \beta = u - b.$$

Ce sont les *variables d'Elsässer*.[4] Ce changement de variables permet de reformuler la MHD en obtenant un système d'équations de transport relativement découplées (avec deux termes de pression non-locaux), soit encore

$$\begin{cases} \partial_t \alpha + (\beta \cdot \nabla)\alpha + \nabla\pi_1 = 0 \\ \partial_t \beta + (\alpha \cdot \nabla)\beta + \nabla\pi_2 = 0 \\ \mathrm{div}(\alpha) = \mathrm{div}(\beta) = 0. \end{cases}$$

[3] Nous faisons ici référence à la théorie des équations de transport dans des espaces de Besov, puisque c'est l'application du calcul paradifférentiel qui permet une borne impliquant la norme Lipschitz $\|\nabla u\|_{L^\infty}$ dans l'exponentielle. Nous renvoyons au Chapitre 3 de [7].

[4] Ces variables sont parfois notées $z^\pm = u \pm b$ par les physiciens ainsi que certains mathématiciens. Notons le fait amusant que "Elsässer" signifie *alsacien* en allemand, d'où le surnom affectueux de *variables alsaciennes* attribué par Franck Sueur.

Dans ces équations, π_1 and π_2 sont deux fonctions *non-nécessairement égales* associées aux deux contraintes *indépendantes* de divergences nulles $\operatorname{div}(\alpha) = 0$ and $\operatorname{div}(\beta) = 0$. La plus grande symétrie du système d'Elsässer et le fait qu'il soit essentiellement constitués d'équations de transport permet la construction de solutions basées dans des espaces L^p avec $p \neq 2$. C'est le fond des travaux de Secchi [96], qui obtient des solutions dans des espaces de Sobolev $W^{m,p}$ avec $m > 1 + d/p$ et $1 < p < +\infty$. Malheureusement, à ce niveau, l'exposant $p = +\infty$ reste hors de portée, ainsi que, par voie de conséquence les solutions (α, β) de régularité $s = 1$.

La limite de la théorie des solutions dans les espaces de Sobolev se trouve dans le fait que le projecteur de Leray (voir (21) dans l'introduction en anglais) n'est pas continu en tant qu'opérateur sur L^∞, ainsi dans l'inégalité stricte $m > 1 + d/p$ qui empêche d'atteindre la régularité critique $s = 1 + d/p$, et *a fortiori* les solutions de régularité $s = 1$. Il nous faut donc d'affranchir de ce cadre, c'est l'interêt des espaces de Besov que nous présentons dans le paragraphe suivant.

Le rôle des espaces de Besov

L'élan principal pour cette partie est la recherche d'un cadre fonctionnel qui soit plus adapté à l'opérateur de projection de Leray $\mathbb{P}$. Les espaces de Besov, qui sont un des outils centraux de nos travaux, sont inspirés d'une remarque très simple : si l'opérateur $\mathbb{P}$ n'est pas borné sur L^∞, en revanche, l'opérateur T_ϕ défini par

$$\forall f \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}^d), \qquad \widehat{T_\phi f}(\xi) = \left(1 - \frac{\xi \otimes \xi}{|\xi|^2}\right) \phi(\xi) \widehat{f}(\xi)$$

est lui bien borné $L^p \longrightarrow L^p$ pour chaque valeur de l'exposant $p \in [1, +\infty]$ à condition que la fonction $\phi \in \mathcal{D}$ ait son support borné loin de $\xi = 0$ et $|\xi| \to +\infty$. En vue de s'éloigner dans le même temps des hautes et des basses fréquences, nous nous munissons d'une partition dyadique de l'unité, c'est à dire une fonction lisse et radiale $\varphi(\xi)$ dont le support soit un anneau de $\mathbb{R}^d$ et telle que

$$1 = \sum_{j \in \mathbb{Z}} \varphi(2^j \xi) := \sum_{j \in \mathbb{Z}} \varphi_j(\xi), \qquad \text{for all } \xi \neq 0.$$

Ensuite, pour toute fonction f, nous définissons l'opérateur $\dot{\Delta}_j$ en posant $\mathcal{F}[\dot{\Delta}_j f](\xi) = \varphi_j(\xi) \widehat{f}(\xi)$. De façon formelle, n'importe quelle fonction f peut être reconstruite à partir des $\dot{\Delta}_k f$ avec l'identité formelle

$$f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f.$$

Cette égalité est la *décomposition de Littlewood-Paley* (homogène) de la fonction f, qui répond assez bien à nos attentes : grâce à ce que nous avons dit sur les opérateurs T_ϕ ci-dessus, nous pouvons affirmer la continuité de $\dot{\Delta}_j \mathbb{P}$ sur tous les espaces L^p avec $p \in [1, +\infty]$, donc avec l'exposant $p = +\infty$ compris ! Par extension, le projecteur de Leray est continu sur l'espace $\dot{B}^s_p$ défini par la (semi) norme *ad hoc*

$$\|f\|_{\dot{B}^s_p} := \sum_{j \in \mathbb{Z}} \|(-\Delta)^{s/2} \dot{\Delta}_j f\|_{L^p}.$$

L'espace $\dot{B}^s_p$ est un premier exemple d'*espace de Besov homogène* et nous donne un petit aperçu de ce qui constitue un des incontournables de cette thèse

Les derniers paragraphes de ce résumé sont dédiés à un survol de nos résultats principaux.

Solutions bornées

La place spéciale de l'exposant $p = +\infty$ justifie l'interêt tout spécial porté aux solutions qui sont bornées, sans aucune hypothèse de décroissance à l'infini $|x| \to +\infty$. Cependant, ce cadre n'est pas dénué de défis, particulièrement du point de vue de l'analyse fonctionnelle (par exemple,

L^∞ est non-séparable et non-réflexif). Par ailleurs, l'absence de décroissance des solutions induit également son lot de problèmes, le plus grand étant que le problème de Cauchy a une infinité de solutions bornées, même sous les hypothèses de régularité les plus fortes.

Afin d'éclaircir ce dernier point, nous nous pencherons sur le cas des équations d'Euler incompressibles (2) qui concentrent toute la difficulté du problème. Nous définissons une solution par
$$u(t,x) = f(t) \qquad \text{and} \qquad \Pi(t,x) = -f'(t)\cdot x,$$
où la fonction $f \in C^\infty(\mathbb{R};\mathbb{R}^d)$ est lisse. En particulier, en prenant f de valeur initial $f(0) = 0$, nous obtenons une infinité de solutions bornées associées à la même condition initiale $u_0 = 0$. Il y a de nombreuses manières d'interpréter cette non-unicité. Le lecteur intéressé consultera le Chapitre 2.

Cette question est intimement liée à la possibilité d'appliquer le projecteur de Leray à des solutions bornées des équations d'Euler. D'une part, il est tout à fait possible d'écrire des équations projetées
$$\partial_t u + \mathbb{P}\operatorname{div}(u\otimes u) = 0 \tag{3}$$
qui ont un sens précis pour un flot borné $u \in C^0(L^\infty)$. Même si le projecteur de Leray n'est pas bien défini comme opérateur sur L^∞, l'opérateur de rang un $\mathbb{P}\operatorname{div} : L^\infty \longrightarrow B^{-1}_{\infty,\infty}$ l'est en revanche. Pak et Park [87] construisent ainsi des solutions Besov-Lipschitz de (3) qui sont uniques à condition initiale $u_0 \in B^1_{\infty,1}$ donnée. D'un autre coté, même si toutes les solutions de (3) sont bien des solutions des équations d'Euler, la réciproque ne peut pas être vraie, sinon les équations d'Euler bénéficieraient de de la propriété d'unicité de [87].

La question que nous posons est la suivante : *À quelle condition une solution des équations d'Euler est-elle également une solution du problème projeté* (3) *?*

Ce problème n'est pas nouveau, et de nombreux résultats existent déjà,[5] principalement sous forme de conditions suffisantes que les solutions d'Euler doivent vérifier pour être solutions de (3). À titre d'exemple, si le flot est borné $u \in C^0(L^\infty)$ et que la pression vérifie l'une des deux conditions
$$\Pi(t) \in \mathrm{BMO} \qquad \text{ou} \qquad \Pi(t,x) \underset{|x|\to+\infty}{=} o(|x|),$$
alors u est automatiquement une solution du problème projeté (3). Nous renvoyons le lecteur aux articles de J. Kato [67] et Kukavica-Vicol [72]. D'autres conditions peuvent être exprimées sur le champ des vitesses, comme dans le livre [77] de Lemarié-Rieusset (Théorème 11.1), l'article [54] de Lemarié-Rieusset and Fernández-Dalgo [54], ou encore notre propre résultat [29]. Le Chapitre 2 contient une plus ample description du problème ainsi qu'une bibliographie sélective, mais plus complète.

Dans le Chapitre 2, nous donnons et démontrons notre résultat principal sur cette question, Théorème 2.2, reproduit ci-dessous. Ainsi formulé, il représente une amélioration de celui donné dans notre article [25].

Théorème 1. *Soit $T > 0$, $u_0 \in L^\infty$ et $u \in C^0([0,T[;L^\infty)$ une solution faible des équations d'Euler* (2) *associée à la donnée initiale u_0 et à la pression $\pi \in \mathcal{D}'([0,T[\times\mathbb{R}^d)$. Les affirmations suivantes sont équivalentes :*

(i) *le flot u est solution du problème projeté avec condition initiale $u(0)$, qui est telle que $u(0) - u_0 \in \mathrm{Cst} \in \mathbb{R}$,*

[5] Tous les travaux dont nous avons la connaissance concernent les équations de Navier-Stokes, qui présentent évidemment les mêmes difficultés. Les résultats que nous énonçons ci-après s'adaptent sans mal au cadre des équations d'Euler (*c.f.* le Chapitre 3 de cette thèse).

(ii) *pour tous temps* $t \in [0, T[$, *nous avons* $u(t) - u(0) \in \mathcal{S}'_h$,

(iii) *pour tous temps* $t \in [0, T[$, *nous avons* $u(t) - u(0) \in \mathrm{BMO}^{-1}$,

(iv) *la pression vérifie* $\pi \in C^0(]0, T[; \mathrm{BMO})$,

(v) *la force de pression est continue en temps* $\nabla \pi \in C^0(]0, T[; \mathcal{S}')$ *et* $\nabla \pi(t) \in \mathcal{S}'_h$ *pour tous temps* $0 < t < T$.

D'une certaine manière, le Théorème 1 est optimal, puisqu'il donne une condition *nécessaire et suffisante* pour qu'une solutions des équations d'Euler soit aussi une solution du problème projeté, exprimée en termes de la pression ou du champ des vitesses. C'est un progrès réel par rapport à plusieurs des résultats précédents, tout en démontrant que certains autres sont en réalité optimaux, [67] par exemple.

La démonstration du Théorème 1 repose sur une combinaison d'analyse de Fourier et d'estimations intégrales. Nous en donnons les grandes lignes ici. La première étape est de montrer que toute solution des équations d'Euler est une solution du problème projeté, *à un polynôme près*. En d'autres termes, il existe une fonction du temps $g : \mathbb{R} \longrightarrow \mathbb{R}^d$ telle que

$$\partial_t u + \mathbb{P}\,\mathrm{div}(u \otimes u) + g(t) = 0,$$

de sorte que u soit une solution de (3) si et seulement si $g \equiv 0$. Puisque, pour chaque temps fixé, $g(t)$ est une fonction constante de l'espace, déterminer si $g(t) = 0$ peut se voir uniquement en regardant les basses fréquences, la transformée de Fourier de $g(t)$ étant un multiple de la masse de Dirac $\delta_0(\xi)$. Nous introduisons donc une fonction de troncature en fréquence $\chi(\xi)$ qui soit égale à $\chi(\xi) = 1$ au voisinage de l'origine $\xi = 0$ et appliquons le multiplicateur de Fourier $\chi(\lambda D)$ à l'équation, avec la perspective d'étudier l'asymptotique des basses fréquences $\lambda \to +\infty$. Nous avons

$$\partial_t \chi(\lambda D) u + \chi(\lambda D)\,\mathrm{div}(u \otimes u) + g(t) = 0.$$

Le coeur de la preuve est de démontrer que le deuxième terme de cette équation disparaît dans la limite. Plus précisément, nous sommes en mesure de démontrer que

$$\big\| \chi(\lambda D) \mathbb{P}\,\mathrm{div}(u \otimes u) \big\|_{L^\infty} = O\left(\frac{1}{\lambda} \log(\lambda) \right) \qquad \text{as } \lambda \to +\infty.$$

Cette estimation, qui semble très intuitive au vu des inégalités de Bernstein (Lemme 1.7), est en réalité plus technique qu'il n'y paraît, le symbole de $\mathbb{P}\,\mathrm{div}$ n'étant pas suffisamment régulier pour que les inégalités de Bernstein s'appliquent. Elle repose plutôt sur un double découpage en fréquence et en espace du noyau de l'opérateur $\mathbb{P}\,\mathrm{div}$. En définitive, ce que nous obtenons est que $g(t)$ est nul exactement quand $\partial_t u$ est élément de l'espace de Chemin $\mathcal{S}'_h$ des distributions dont la transformée de Fourier possède une forme de décroissance à $\xi = 0$. L'énoncé du Théorème 1 est simplement une version intégrée de cette affirmation.

Variables d'Elsässer et résolution du problème de Cauchy

Nous décrivons maintenant comment nous appliquons les techniques décrites ci-dessus à la résolution du problème de Cauchy de la MHD idéale. En quelque sorte, il s'agit là du noyau dur de ce travail de thèse. Nous nous penchons sur la MHD idéale écrite en variables d'Elsässer :

$$\begin{cases} \partial_t \alpha + (\beta \cdot \nabla)\alpha + \nabla \pi_1 = 0 \\ \partial_t \beta + (\alpha \cdot \nabla)\beta + \nabla \pi_2 = 0 \\ \mathrm{div}(\alpha) = \mathrm{div}(\beta) = 0. \end{cases} \tag{4}$$

Bien-entendu, les solutions bornées de ce système posent les mêmes problèmes que ceux décrits dans les paragraphes précédents pour les équations d'Euler. Les solutions ne sont pas uniques à moins qu'une condition soit imposée sur la pression ou le champ des vitesses. Mais avec le système plus complexe de la MHD, une difficulté nouvelle apparaît : même si le système (4) d'Elsässer est obtenu à partir de la MHD (1) par un simple changement de variable, les deux systèmes *ne sont pas en fait équivalents*. Même si toute solution (u, b) du système "de base" (1) définit bien une solution (α, β) de (4), la réciproque est fausse. En fait, en prenant la somme et la différence des deux première équations de (4) donne

$$\partial_t b + \operatorname{div}(u \otimes b - b \otimes u) = \frac{1}{2}\nabla(\pi_2 - \pi_1) = f'(t),$$

de sorte que $(u, b) = \frac{1}{2}(\alpha + \beta, \alpha - \beta)$ ne peut être une solution de la MHD (1) que si $\nabla\pi_1 = \nabla\pi_2$. À nouveau, un contre-exemple d'une grande simplicité montre que cela n'a rien d'hypothétique :

$$\alpha(t, x) = f(t) = -\beta(t, x), \qquad \pi_1(t, x) = -f'(t) \cdot x = \pi_2(t, x).$$

Alors $(u, b) = \frac{1}{2}(\alpha + \beta, \alpha - \beta)$ n'est pas une solution de (1) dès que $f'(t) \neq 0$.

Le Théorème 3.11 contient une condition nécessaire et suffisante d'équivalence entre les deux systèmes. De plus, en généralisant les méthodes déployées au Chapitre 2 pour les équations d'Euler, nous pouvons appliquer le projecteur de Leray à l'équation des moments, pourvu que le champ des vitesses satisfasse la condition de basse fréquence $u(t) - u(0) \in \mathcal{S}'_h$. Dans ce cas, nous obtenons les deux systèmes équivalents suivants :

$$(5) \qquad \begin{cases} \partial_t u + \mathbb{P}\operatorname{div}(u \otimes u - b \otimes b) = 0 \\ \partial_t b + \operatorname{div}(u \otimes b - b \otimes u) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} \partial_t \alpha + \mathbb{P}\operatorname{div}(\beta \otimes \alpha) = 0 \\ \partial_t \beta + \mathbb{P}\operatorname{div}(\alpha \otimes \beta) = 0. \end{cases}$$

Les relations entre tous les différents systèmes que nous avons évoqués sont résumées dans le diagramme suivant.

$$\begin{tikzcd}
\text{Els} \arrow[r, "\alpha,\beta"] \arrow[d, "b"] & \mathbb{P}(\text{Els}) \arrow[l, bend left] \arrow[d, shift right] \\
\text{MHD} \arrow[r, "u"] \arrow[u, bend left] & \mathbb{P}(\text{MHD}) \arrow[l, bend left] \arrow[u, shift right]
\end{tikzcd}$$

Ici, Els représente la MHD en variables d'Elsässer (4) et le sigle MHD représente le système "de base" (1). Les symboles $\mathbb{P}(\text{MHD})$ et $\mathbb{P}(\text{Els})$ font référence, respectivement, aux deux systèmes projetés de (5). Les flèches représentent les conditions nécessaires pour passer d'un système à l'autre, et une flèche seule indique qu'aucune condition n'est nécessaire.

La MHD idéale à laquelle est appliquée la projection de Leray et formulée en variables d'Elsässer est en fait un système d'équations de transport avec comme second membres des commutateurs : nous pouvons écrire

$$\begin{cases} \partial_t \alpha + (\beta \cdot \nabla)\alpha = \big[\beta \cdot \nabla, \mathbb{P}\big]\alpha \\ \partial_t \beta + (\alpha \cdot \nabla)\beta = \big[\alpha \cdot \nabla, \mathbb{P}\big]\beta. \end{cases}$$

Résoudre le problème de Cauchy dans des espaces de Besov passe par l'utilisation d'outils habituels d'analyse de Littlewood-Paley : application d'un bloc dyadique, estimations de commutateurs passant par la décomposition en paraproduit de Bony et sommation pour obtenir des normes Besov. Sans donner plus de détail quant à la preuve, présentons simplement le résultat.

Théorème 2. *Soit $p, r \in [1, +\infty]$ et $s \in \mathbb{R}$ tels que l'espace de Besov $B^s_{p,r}$ soit sous-critique : $B^s_{p,r} \subset W^{1,\infty}$ ou en d'autres termes*

$$s > 1 + \frac{d}{p} \qquad \text{ou} \qquad s = 1 + \frac{d}{p} \text{ et } r = 1.$$

Considérons des données initiales de divergences nulles $u_0, b_0 \in B^s_{p,r}$. Il existe un temps $T > 0$ tel que le système (3.1) possède une unique solution $(u, b) \in C^0([0, T[; B^s_{p,r})$ si $r < +\infty$, et[6] dans l'espace $C^0_w([0, T[; B^s_{p,r})$ si $r = +\infty$, associée aux données initiales (u_0, b_0) telle que, si $p = +\infty$,

$$u(t) - u(0) \in \mathcal{S}'_h.$$

De plus, si $r < +\infty$, nous avons aussi $u, b \in C^1([0, T[; B^{s-1}_{p,r})$ et $\partial_t u, \partial_t b \in C^0_w([0, T[; B^{s-1}_{p,r})$ quand $r = +\infty$. Enfin, le temps T peut être choisi de telle manière à ce que

$$T \geq \frac{C}{\|u_0, b_0\|_{B^s_{p,r}}},$$

pour une certaine constante $C = C(d, s, p, r)$.

La résolution du problème de Cauchy dans des espaces de Besov n'est pas, en elle-même un résultat nouveau : elle a déjà été faite par Miao et Yuan [83]. Cependant, nos propres résultats [28], [29], [25] contribuent au sujet de deux manières. Premièrement, les estimations *a priori* que nous proposons constituent une très grande simplification par rapport aux calculs complexes de [83] basés sur la composition par le flot différentiel de $u \pm b$. Par ailleurs, notre résultat permet de complètement clarifier la signification de l'unicité dans le cadre des solutions bornées de la MHD, le résultat de [83] ne portant que sur le système projeté (5).

Parmi les autres résultats que nous exposons dans la thèse se trouve un certain nombre de critères de continuation des solutions. Par exemple, la solution $(\alpha, \beta) \in C^0([0, T[; B^s_{p,r})$ du Théorème 2 peut être prolongée en une solution de même régularité au delà du temps T si et seulement si (voir Proposition 3.22)

$$\int_0^T \left\{ \|\Delta_{-1}(u, b)\|_{L^\infty} + \|\nabla(u, b)\|_{L^\infty} \right\} \mathrm{d}t < +\infty. \tag{6}$$

Ici, Δ_{-1} est le bloc de Littlewood-Paley non-homogène associé aux basses fréquences. Ce résultat, analogue à celui de Beale-Kato-Majda [8], doit être comparé à ceux similaires déjà obtenus pour la MHD : celui de Caflish, Klapper et Steele [16] portant sur $\|(\omega, j)\|_{L^\infty}$, ou les résultats plus récents [104], [18] ou [23] qui améliorent la régularité requise de ∇u et ∇b (le Chapitre 3 contient, en plus de références, quelques explications). La nouveauté de notre traitement (6) ne se situe pas dans l'utilisation d'un meilleur espace fonctionnel, mais plutôt dans le fait que le critère (6) reste valable *pour des solutions bornées* des équations projetées. Nous proposons, dans le Chapitre 3, diverses améliorations. Par exemple, une forme d'interpolation logarithmique montre que la régularité requise en temps des basses fréquences peut être grandement allégée : l'intégrale (6) peut être remplacée par

$$\int_0^T \|(\nabla u, \nabla b)\|_{L^\infty} \left\{ 1 + \log\left(1 + \frac{\|\Delta_{-1}(u, b)\|_{L^\infty}}{\|(\nabla u, \nabla b)\|_{L^\infty}} \right) \right\} \mathrm{d}t < +\infty.$$

Nous renvoyons au Chapitre 3 pour d'autres critères utilisant la structure particulière des équations afin de détecter une explosion potentielle sur une seule de quantités $\nabla^2 u$ ou $\omega \pm j$.

[6] Le symbole $C^0_w([0, T[; X)$ fait référence à l'espace des fonctions $f : [0, T[\longrightarrow X$ à valeurs dans un espace de Banach X de prédual Y qui soient continues pour la topologie faible : pour toute $\phi \in Y$, le crochet $t \mapsto \langle f(t), \phi \rangle_{X \times Y}$ est une fonction continue sur $[0, T[$.

Amélioration du temps de vie des solutions

Ensuite, un des axes de nos recherches est l'obtention de bornes inférieures pour le temps de vie des solutions qui tiennent compte de la proximité ou non de la MHD avec un système globalement bien posé. En particulier, nous étudions le cas de la dimension $d = 2$ et le régime des faibles champs magnétiques. L'idée générale est que si $b \to 0$, la MHD tend formellement vers les équations d'Euler 2D, qui ont d'uniques solutions globales. Nous cherchons donc à montrer que, dans ce régime, le temps de vie des solutions peut devenir arbitrairement grand.

Nous avons développé deux méthodes pour ce faire. La première, exposée à la fin du Chapitre 3 (Théorème 3.29), utilise une mesure de la différence entre la MHD et les équations d'Euler qui respecte les symétries du système mises en évidence par les variables d'Elsässer. La deuxième, présentée au Chapitre 4, utilise de manière extensive le fait que pour dans le cas $p = +\infty$ de solutions $\alpha, \beta \in C^0_T(B^1_{\infty,1})$ nous disposons de deux quantités

$$X = \mathrm{curl}\,(\alpha) = \omega + j \qquad \text{and} \qquad Y = \mathrm{curl}\,(\beta) = \omega - j.$$

d'ordre zero, puisque appartenant à l'espace $C^0_T(B^0_{\infty,1})$. Rappelons qu'atteindre ce contexte était l'un des buts de notre démarche. Les "tourbillons" X et Y sont solutions d'équations de transport

$$\begin{cases} \big(\partial_t + \beta \cdot \nabla\big)X = \mathcal{L}(\nabla\alpha, \nabla\beta) \\ \big(\partial_t + \alpha \cdot \nabla\big)Y = \mathcal{L}(\nabla\beta, \nabla\alpha) \end{cases}$$

dont les second membres $\mathcal{L}(\nabla\alpha, \nabla\beta)$ sont des matrices dépendant de façon bilinéaire des coefficients de $\nabla\alpha$ and $\nabla\beta$ et sont donnés par

$$\big[\mathcal{L}(\nabla\alpha, \nabla\beta)\big]_{ij} = \partial_j\beta_k\partial_k\alpha_i - \partial_i\beta_k\partial_k\alpha_j.$$

Une étude attentive de ce terme $\mathcal{L}(\nabla\alpha, \nabla\beta)$ dans le régime des faibles champs magnétiques, une utilisation fine de la décomposition de Bony (rappelons que l'espace $B^0_{\infty,1}$ n'est pas une algèbre de Banach) et le recours à des estimations linéaires pour l'équation de transport dans l'espace de régularité zero $B^0_{\infty,1}$ conduisent à une inégalité triplement logarithmique pour le temps de vie (Théorème 4.10),

$$T \geq \frac{C}{\|(u_0, b_0)\|_{B^2_{\infty,1}\cap L^2}} \log\left\{1 + \log\left[1 + \log\left(1 + C\frac{\|(u_0, b_0)\|_{B^1_{\infty,1}\cap L^2}}{\|b_0\|_{B^1_{\infty,1}}}\right)\right]\right\}.$$

En dernier lieu, signalons que la fin du Chapitre 4 contient également une exploration des gradients symétriques

$$\big[\nabla_\sigma\alpha\big]_{ij} = \frac{1}{2}\Big(\partial_i\alpha_j + \partial_j\alpha_i\Big)$$

and $\nabla_\sigma\beta$. Nous établissons les équations satisfaites par ces quantités et les appliquons au cas particulier des solutions linéaires par rapport à la variable d'espace

$$\alpha(t, x) = A(t).x \qquad \text{and} \qquad \beta(t, x) = B(t).x,$$

où $A(t), B(t) \in C^1(\mathbb{R}^d \otimes \mathbb{R}^d)$ sont des matrices dépendantes du temps. Ces solutions particulières, présentes dans [81], ont été étudiées pour les équations d'Euler et de Navier-Stokes 3D dans [84] en vue d'en étudier l'explosion en temps fini. Dans notre cas, nous trouvons un système différentiel pour $A(t)$ et $B(t)$ qui est *linéaire*, et a donc des solutions globales.

MHD en rotation rapide

Dans la dernière partie de la thèse, nous étudions un modèle de MHD incompressible à densité variable en rotation rapide. L'objectif est de compléter nos travaux précédents [29] sur la question.[7] Du point de vue de la physique, il s'agit de comprendre la dynamique des fluides géophysiques, qui sont caractérisés par la prépondérance de de la force de Coriolis par rapport aux autres effets.

Le terme de *fluide géophysique* fait référence à un fluide de grande échelle se mouvant à la surface d'un corps céleste en rotation autour de son axe, comme par exemple les flots atmosphériques ou océaniques, le magma terrestre ou les fluides stellaires. L'importance des effets dus à la rotation, par l'intermédiaire de la force d'inertie de Coriolis, est décrite par un nombre sans dimension, le *nombre de Rossby Ro* qui encode le rapport entre la cinétique du fluide et la force de Coriolis. De manière schématiques, le nombre de Rossby est inversement proportionnel à la vitesse de rotation du référentiel dans lequel évolue le fluide.

Les équations primitives qui sont à la base de notre étude seront celles de la MHD incompressible 2D à densité variable $\rho(t,x) \geq 0$. Ces équations s'écrivent

$$\begin{cases} \rho\,\partial_t u + \rho(u\cdot\nabla)u + \frac{1}{\epsilon}\nabla\Pi + \frac{1}{\epsilon}\rho u^\perp = h(\epsilon)\,\mathrm{div}(\nu(\rho)\nabla u) + (b\cdot\nabla)b - \frac{1}{2}\nabla(|b|^2) \\ \partial_t b + (u\cdot\nabla)b = (b\cdot\nabla)u + h(\epsilon)\nabla^\perp(\mu(\rho)\mathrm{curl}\,(b)) \\ \partial_t\rho + u\cdot\nabla\rho = 0 \\ \mathrm{div}(u) = 0. \end{cases} \tag{7}$$

Dans le cadre de notre travail, le nombre de Rossby est noté $Ro = \epsilon$, ce qui introduit implicitement la trajectoire que nous allons suivre : étudier l'asymptotique $\epsilon \to 0^+$. Dans les équations qui précédent, les quantités scalaires $h(\epsilon)\nu(\rho)$ et $h(\epsilon)\mu(\rho)$ sont respectivement la viscosité et la résistivité du fluides, que nous autorisons à dépendre de la densité et du paramètre de rotation. Le terme $\epsilon^{-1}\rho u^\perp$ correspond à la force de Coriolis.

Le cadre de la dimension deux et des fluides incompressibles peut paraître, à première vue, quelque peu restrictif. Cependant, de nombreuses raisons militent en faveur de ces choix. Premièrement, en termes strictement techniques, étudier la limite en rotation rapide d'un système de la forme (7) dans le cas général de trois dimensions est, à ce jour, largement hors de notre portée. Ensuite, un certain nombre de propriétés connues des fluides géophysiques permettent de souligner la pertinence de nos hypothèses. Ainsi, une des caractéristiques de ces fluides est la quasi-incompressibilité et le fait que l'évolution se limite à des mouvements horizontaux. Ce principe est connu sous le nom de théorème de Taylor-Proudman et a été rigoureusement démontré à partir de plusieurs modèles. Par exemple, nous pouvons citer les travaux de Feireisl, Gallagher et Novotný [50] concernant les fluides compressibles et le livre [21] que le lecteur intéressé pourra consulter.

Dans notre premier travail [27], nous avons étudié la limite $\epsilon \to 0^+$ du système (7) dans le cas dissipatif $h(\epsilon) = 1$ et avec des données initiales *mal préparées* : en d'autres termes, les données initiales (ρ_0, u_0, b_0) pouvaient dépendre de ϵ tant que les bornes suivantes étaient respectées :

$$(b_{0,\epsilon})_{\epsilon>0} \subset L^2, \qquad (m_{0,\epsilon})_{\epsilon>0} \subset L^2, \qquad \left(\frac{|m_{0,\epsilon}|^2}{\rho_{0,\epsilon}}\right)_{\epsilon>0} \subset L^1, \qquad (\rho_{0,\epsilon})_{\epsilon>0} \subset L^\infty.$$

Dans ce qui précède, la notation $(f_\epsilon)_\epsilon \subset X$ signifie que la suite de fonctions (f_ϵ) est bornée dans l'espace de Banach X et $m = \rho u$ est la quantité de mouvement du fluide. L'étape suivante est de se servir de ces propriétés et de la conservation de l'énergie totale pour en déduire des propriétés de convergence faible des solutions $(\rho_\epsilon, u_\epsilon, b_\epsilon)$ vers des solutions d'un système limite d'EDP. Ce

[7] Les résultats obtenus dans [29] ont fait l'objet d'un mémoire de master et ne sont pas reproduits ici.

procédé utilise de façon extensive la théorie de Di Perna-Lions et des méthodes de compacité par compensation héritées de [55] et [46].

Un cas spécial d'intérêt tout particulier est celui des fluides quasi-homogènes, où la densité initiale $\rho_{0,\epsilon}$ est supposée être une perturbation d'un état constant

$$\rho_{0,\epsilon} = 1 + \epsilon r_{0,\epsilon}, \qquad \text{avec } (r_{0,\epsilon})_{\epsilon>0} \subset L^\infty \cap L^2.$$

Dans ce cas, le système limite est un modèle de *MHD quasi-homogène*, à savoir

$$\begin{cases} \partial_t U + (U\cdot\nabla)U + RU^\perp + \nabla\left(\Pi + \frac{1}{2}|B|^2\right) = (B\cdot\nabla)B + \nu(1)\Delta U \\ \partial_t B + (U\cdot\nabla)B = (B\cdot\nabla)U + \mu(1)\Delta B \\ \partial_t R + U\cdot\nabla R = 0 \\ \mathrm{div}(U) = 0. \end{cases} \tag{8}$$

Les fonctions (R, U, B) étant les limites (faibles) de $(r_\epsilon, u_\epsilon, b_\epsilon) \rightharpoonup (R, U, B)$.

Dans le Chapitre 5 de la thèse, nous prolongeons cette étude en donnant une estimation explicite de la vitesse de convergence, sous hypothèse de convergence forte des données initiales. Définissons les quantités

$$\delta r_\varepsilon = r_\epsilon - R, \qquad \delta u_\varepsilon = u_\epsilon - U, \qquad \delta b_\varepsilon = b_\epsilon - B.$$

Alors, en faisant essentiellement la différence entre le système limite (8) et les équations primitives (7), nous pouvons démontrer un théorème de structure des solutions $(r_\epsilon, u_\epsilon, b_\epsilon)$ à toute valeur du paramètre $\epsilon > 0$. Elles sont égales à la somme de la limite (R, U, B) et d'un reste de taille $O(\epsilon)$, si μ, ν sont, par exemple, de classe C^1. Ce résultat est consigné dans le Théorème 5.4, reproduit ci-dessous.

Théorème 3. *Soit $h \equiv 1$ dans (7) et ν, μ des fonctions continues minorées par un constante strictement positive $\min(\mu(\rho), \nu(\rho)) \geq \eta > 0$. Pour tout module de continuité fixé σ, nous faisons l'hypothèse supplémentaire que $\nu, \mu \in C_\sigma(\mathbb{R})$. Soit encore une suite de données initiales $\big(\rho_{0,\varepsilon}, u_{0,\varepsilon}, b_{0,\varepsilon}\big)_{\varepsilon>0}$ remplissant les conditions de la Sous-section 5.2.1, et soit $\big(\rho_\varepsilon, u_\varepsilon, b_\varepsilon\big)_{\varepsilon>0}$ une suite de solutions faibles d'énergie finie de (7) correspondant à ces données initiales. Soit $M > 0$ défini par*

$$M := \sup_{\varepsilon>0} \|r_{0,\varepsilon}\|_{L^\infty} + \sup_{\varepsilon>0} \|u_{0,\varepsilon}\|_{L^2} + \sup_{\varepsilon>0} \|b_{0,\varepsilon}\|_{L^2}.$$

Supposons encore que le triplet $\big(R_0, U_0, B_0\big)$, défini en (5.6), appartienne à l'espace $H^{1+\beta}(\Omega) \times H^1(\Omega) \times H^1(\Omega)$, pour un certain $\beta \in]0,1[$, et soit $\big(R, U, B\big)$ l'unique solution du système (8), donnée par le Théorème 5.3. Enfin, soit

$$\delta r_\varepsilon := r_\varepsilon - R\,, \qquad \delta u_\varepsilon := u_\varepsilon - U\,, \qquad \delta b_\varepsilon := b_\varepsilon - B$$

et $\delta r_{0,\varepsilon} := r_{0,\varepsilon} - R_0$, $\delta u_{0,\varepsilon} := u_{0,\varepsilon} - U_0$ et $\delta b_{0,\varepsilon} := b_{0,\varepsilon} - B_0$ définis de façon analogue.

Alors, pour tout temps $T > 0$, nous avons les estimations suivantes : pour toute valeur du paramètre $\varepsilon > 0$ et presque tout temps $t \in [0, T]$,

$$\begin{aligned} &\|\delta r_\varepsilon(t)\|_{L^2}^2 + \|\delta u_\varepsilon(t)\|_{L^2}^2 + \|\delta b_\varepsilon(t)\|_{L^2}^2 + \int_0^t \left\{ \|\nabla \delta u_\varepsilon\|_{L^2}^2 + \|\nabla \times \delta b_\varepsilon\|_{L^2}^2 \right\} \mathrm{d}\tau \\ &\qquad \leq C \left\{ \|\delta r_{0,\varepsilon}\|_{L^2}^2 + \|\delta u_{0,\varepsilon}\|_{L^2}^2 + \|\delta b_{0,\varepsilon}\|_{L^2}^2 + \max\left\{\varepsilon^2\,,\, \sigma^2(M\varepsilon)\right\} \right\}, \end{aligned} \tag{9}$$

où la constante $C > 0$ ne dépend que de T, de η, des quantités $|\nu|_{\mathcal{C}_\sigma}$ et $|\mu|_{\mathcal{C}_\sigma}$, des normes des données initiales $\|u_0\|_{H^1}$, $\|b_0\|_{H^1}$ and $\|r_0\|_{H^{1+\beta}}$, et de M.

La difficulté principale dans la démonstration de ce théorème réside dans le fait que prendre naïvement la différence entre les systèmes (7) et (8) introduit des objets de régularité très basse. Nous avons donc élaboré une inégalité d'entropie relative afin de rendre le principe rigoureux, en suivant les idées de Feireisl, Jin et Novotný [51].

Signalons que la résolution du problème de Cauchy donnée au Chapitre 3 de la thèse permettent aussi d'utiliser cette inégalité d'entropie relative afin d'obtenir la convergence dans le cas de viscosités évanescentes $h(\epsilon) \longrightarrow 0^+$. C'est l'objet du Théorème 5.7.

Notons également que l'ensemble de nos résultats concernant le problème de Cauchy de la MHD idéale peuvent s'adapter aux équations (8) sans dissipation $h(0) = 0$. La MHD quasi-homogène est en fait le cadre originel de nos travaux, et est celui de nos articles [28], [29].

Introduction

There's nothing like deduction. We've determined everything about our problem but the solution.
Isaac Asimov[8]

Forward

This PhD dissertation is the outcome of three years of research on the mathematics of magnetohydrodynamics (MHD for short). As can be told by the name, MHD deals with fluids in interaction with a magnetic field because of their ability to conduct electricity. The topic has been developed as a part of plasma physics as early as the 1940s with the work of Hannes Alfvén, and has been studied by mathematicians as early as the late 1980s.

The reader will have understood that MHD is by no means new in the scientific landscape. Despite the extraordinary diversity of applications, interesting features or open questions, many problems in MHD have already received intense attention, and the ones we explore in this dissertation are no exception. For this reason, although we do our best to provide an accurate representation of current research, our bibliography cannot have the pretense to be complete.

One of the factors which has largely contributed to the abundance of literature is that the MHD equations have sometimes been treated as a mere generalization of the Euler or Navier-Stokes equations, so that there is a strong temptation to study MHD by simply adapting existing results for the Euler or Navier-Stokes systems to MHD. This approach has the unfortunate effect of inciting to neglect the very rich physics of the system: MHD possesses its own interesting and specific features (such as additional conserved quantities, non-trivial static solutions, peculiar stability issues, a much more intricate well-posedness theory, *etc.*).

Though we do not have the pretense to be completely immune to this philosophy of MHD, this remark is meant to throw a light on the goal of this dissertation. We aim at providing a deeper understanding of the MHD equations by highlighting the specific structure inherent to the system and the methods that are tailored to them. This will be done by sometimes revisiting existing results and providing improvements, and sometimes by presenting our own work on the topic.

In the remainder of this introduction, we will provide some quick background concerning the ideal MHD system, which will be the main topic of the dissertation. After presenting the broad context in which these equations apply and appear, we have chosen to sort our remarks by discussing first on a physical perspective, and then a mathematical one. The distinction between these two points of view is quite arbitrary, especially on the very formal level of this introduction: to clarify, we reserve the issues related to functional analysis or well-posedness for a more "mathematical" paragraph, while features of MHD inherited from other physical systems (such as the Maxwell equations or Newtonian physics) will be classified as "physical".

In order to accommodate a hurried reader, the Introduction ends with a summary of the

[8] Isaac Asimov, *Runaround.*

dissertation, at page 40, which is designed to be relatively independent from the rest of the introduction and contains our main results.

Context: Conducting Fluids

The general frame in which MHD was born is plasma physics.[9] A plasma is a fluid in an ionized state, and which can therefore conduct electrical current. This changes massively the dynamics of the fluid: the electrical current induces a magnetic field, which acts on the fluid *via* Laplace force, thus changing the dynamics of the fluid and the currents. As a consequence of this double interaction between the fluid and the magnetic field, any appropriate description of a plasma must involve a non-linear coupling between a kinetic or a fluid equation (such as the Navier-Stokes equations or a Vlasov equation) and the Maxwell equations.

If plasmas are not common at the human scale, there are numerous examples of these fluids if one looks further: 90% of visible matter is estimated to be plasma. This covers stellar fluids (high pressure and temperature matter undergoing fusion), astrophysical plasmas such as the solar wind, or fusion confinement experiments.

However, plasmas are not, strictly speaking, the only physical systems in the scope of MHD. For instance, molten metal (*e.g.* Earth's core) or certain electrolytics will also be accurately described by MHD: they are conducting fluids that are affected by a self-generated magnetic field, or in other words, a *magnetofluid*.

In order to navigate between the many types of magnetofluids, a number of dimensionless parameters may be introduced, which will roughly characterize the fluid and indicate what type of model should be used. For instance, we may define the usual (viscous) Reynolds number Re, an analogous magnetic Reynolds number Rm which quantifies the proportion of energy lost by Joule effect, the magnetic Prandtl number Pm (the ratio between viscosity and resistivity), but also the β number which discriminates between pressure driven or magnetic driven fluids, the Mach number Ma, the Alfvén Mach number, *etc.*

In the sequel, we will be interested in incompressible MHD. In other words, we will assume a set of simplifying assumptions:

1. the fluid is non-relativistic: the speed of the fluid is small when compared to the speed of light so that the kinetics are Newtonian;

2. the fluid is globally neutral and dominated by its magnetic field: this will allow us to perform a quasi-static approximation in the Maxwell equations and neglect the displacement current;

3. the fluid is highly collisional: the dynamics may be described by a fluid equation (instead of a Vlasov or Boltzmann equation).

The number of simplifications we will work with make of incompressible MHD a *low level* description of plasmas. In particular, hot or diluted plasmas (such as stellar fluids) would indeed be more accurately described by a kinetic equation (Vlasov or Boltzmann) that is a scaling limit of a particle system. However, MHD is often much simpler to work with, for instance to investigate stability issues of plasma configurations, and is commonly used as the basis for a subsequent higher level analysis.

[9] We refer to the introductions of [11] and [60] for a more detailed discussion on the history of plasma physics, the definition of a plasma and many examples. Most of the material in this short overview comes from these references.

The Physics of MHD

In this Section, we start our Physical discussion of the ideal MHD equations and their main features. They are a system of PDEs that describe the evolution of the velocity field $u(t,x) \in \mathbb{R}^3$ of the fluid and the magnetic field $b(t,x) \in \mathbb{R}^3$ it generates, evaluated at time $t \in \mathbb{R}$ and position x. The hydrodynamic pressure will be noted $\Pi(t,x) \in \mathbb{R}$. The equations, which are set in the whole space $x \in \mathbb{R}^3$, read

$$\begin{cases} \rho\, \partial_t u + \rho(u \cdot \nabla)u + \nabla\Pi = (b \cdot \nabla)b - \frac{1}{2}\nabla(|b|^2) \\ \partial_t b - \nabla \times (u \times b) = 0 \\ \mathrm{div}(u) = \mathrm{div}(b) = 0. \end{cases} \tag{10}$$

Some further clarifications are required. Firstly, we have noted ρ the density of the fluid, although in most of what follows we will assume the density to be constant and fix its value to $\rho \equiv 1$ for the sake of simplicity. Secondly, the equations above are formulated in the so-called *electromagnetic units* $\mu_0 = 1$ and $\epsilon_0 = c^{-2}$, which are, as we will explain below, particularly appropriate for the context of MHD (see also Section 3 pp. 779-782 in the Appendix of [65]).

The ideal MHD equations (10) can be seen to be a generalization of incompressible fluid mechanics: if there is no magnetic field $b \equiv 0$, then system (10) reduces to the Euler equations. In fact, as we will see, ideal MHD shares a number of interesting properties with the Euler system. For example, it is scaling invariant with respect to both time and space: the equations remain unchanged under independent rescaling of both time and space variables

$$t' = \lambda t \quad \text{and} \quad x' = \mu x \qquad \text{for all } \lambda, \mu > 0,$$

with appropriate rescaling of the derivative operators: $\partial_{t'} = \lambda \partial_t$ and $\nabla_{x'} = \mu \nabla_x$. A very important consequence regarding our system is that it is physically relevant for magnetofluids in all time and space scales, from a human scale tokamak to galactic plasmas (see paragraph 4.1.2 pp. 138-139 in [60]). This nice property is due to the fact that we deal with *ideal* fluids, and is lost for Navier-Stokes or resistive/viscous MHD.

Incompressible Hydrodynamics

One of the most important properties of the magnetofluids considered in this dissertation is that of incompressibility, which is not in fact a specific feature of MHD, but already has strong bearing for Euler or Navier-Stokes equations. Therefore, in order to simplify the discussion, we will assume that $b \equiv 0$ in this paragraph. The incompressibility of the fluid is expressed though the divergence condition $\mathrm{div}(u) = 0$ which is equivalent to the flow $\phi_t(x)$, defined by

$$\partial_t \phi_t(x) = u\big(t, \phi_t(x)\big),$$

being measure conserving: if $\Omega \subset \mathbb{R}^3$, then $|\phi_t(\Omega)| = |\Omega|$.

From a physical perspective, incompressibility introduces a number of singular properties. The conservation of volume implies that the fluid must exhibit a form of *non-local* behavior: for example, if an incompressible fluid lying in a cylinder is pressured on one side of the cylinder, then the pressure must immediately be felt on the other side.

Because of this non-local behavior, information may propagate at infinite speed within the fluid. Another way of saying this is that the speed of sound is infinite inside an incompressible fluid: sound waves in a compressible fluid are carried by density and pressure variations which evolve according to a state law $\Pi = F(\rho)$ and the continuity equation

$$\partial_t \rho + \mathrm{div}(\rho u) = 0.$$

Then if we assume, for example, that the density is constant $\rho \equiv 1$ as in the ideal MHD system (10) above, we eliminate the possibility for density waves to propagate as we introduce non-locality in the problem. But this is not without further consequence on the equations: the compressible Euler equations

$$\begin{cases} \rho\,\partial_t u + \rho\,(u \cdot \nabla)u + \nabla F(\rho) = 0 \\ \partial_t \rho + \operatorname{div}(\rho u) = 0 \end{cases}$$

form a set of four equations for the four unknowns (ρ, u_1, u_2, u_3), so that eliminating the density variable and enforcing incompressibility $\operatorname{div}(u) = 0$ makes the system formally overdetermined. This tension is resolved by noting that, as we evacuate the possibility of density variations $\rho \equiv 1$, we also make the state equation $\Pi = F(\rho)$ of the fluid degenerate: the pressure in an incompressible fluid cannot be given as a function of the density. In fact, the pressure Π in the incompressible Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla\Pi = 0 \\ \operatorname{div}(u) = 0 \end{cases} \tag{11}$$

is an independent unknown in its own right, which is consistent with the fact that system (11) has 4 equations corresponding to the four unknowns (Π, u_1, u_2, u_3). Thus, it has sometimes been noted that the pressure can be seen as a Lagrange multiplier associated to the incompressibility constraint.

This counting of the number of unknowns and equations to see if the system if formally overdetermined or underdetermined will be recurring in our discussion of MHD. In fact, and this will have important consequences, the reader may already check that the ideal MHD system (10) if formally overdetermined.

Maxwell Equations

We set our attention on the electromagnetic part of MHD. The basis for our discussion will be the well-known Maxwell equations, which read, in an inertial reference frame and SI units,[10]

$$\begin{aligned} &\partial_t b = -\nabla \times e && \nabla \times b = \mu_0\, j + c^{-2}\partial_t e \\ &\operatorname{div}(e) = \epsilon_0^{-1}\varrho && \operatorname{div}(b) = 0, \end{aligned} \tag{12}$$

where $e(t,x), b(t,x) \in \mathbb{R}^3$ are the electric and magnetic fields, $\varrho(t,x) \in \mathbb{R}$ is the electrical charge density and $j(t,x) \in \mathbb{R}^3$ is the electrical current vector. It should immediately be noted that the bottom divergence equations are set apart, as they do not play any role in the dynamics of the electromagnetic field: the Maxwell equations (12) only have six unknowns $(e_k, b_k)_{k=1,2,3}$ so that the evolution of (e, b) should be entirely determined by the top six evolution equations.

In fact, the two divergence equations could be dispensed with, as they are contained in the top two of (12) and their initial data. On the one hand, by taking the divergence of the Maxwell-Faraday law, we get $\partial_t \operatorname{div}(b) = 0$, so that the divergence of the magnetic field equation is independent of time. If $\operatorname{div}(b_{t=0}) = 0$ initially, then the condition holds for all times. Similarly, the divergence of the Maxwell-Ampère equation is in fact a conservation law[11]

$$\partial_t \operatorname{div}(e) + \frac{1}{\epsilon_0}\operatorname{div}(j) = 0,$$

[10]The SI units in electromagnetism were historically defined by the following choices: $\mu_0 = 4\pi 10^{-7}\, N \cdot A^{-2}$, $\epsilon_0 = \mu_0^{-1} c^{-2}$ are the magnetic permeability and electric permittivity of vacuum, and $c = 299792458\, m \cdot s^{-1}$ is the speed of light. Nowadays, they are instead defined by more fundamental contants linked to the Standard Model of particule physics.

[11]A conservation law is a PDE of the form $\partial_t f + \operatorname{div}(g) = 0$. By integrating on a domain, it implies that g can be seen as a current vector of the quantity f.

or in other words, $\operatorname{div}(e)$ must be a charge (up to the initial datum) associated with the electrical current j. This is quite a contrast with the divergence condition in the incompressible Euler equations (11), which was necessary to determine the dynamics because of the presence of the pressure Π as an independent unknown.

This discussion of the Maxwell equations sheds light on the two divergence conditions in the ideal MHD equations (10). There is on the one hand the incompressibility constraint $\operatorname{div}(u) = 0$ which is necessary to the dynamics, and on the other the no-flux condition $\operatorname{div}(b) = 0$ which is superfluous for the dynamics because the divergence of the second equation in (10) implies that $\operatorname{div}(b)$ is independent of time,

$$\partial_t \operatorname{div}(b) - \operatorname{div} \nabla \times (u \times b) = \partial_t \operatorname{div}(b) = 0, \tag{13}$$

so that it is enough to enforce it on the initial datum $\operatorname{div}(b_{t=0}) = 0$. This explains why the ideal MHD system (10) is apparently overdetermined.

Now, to apply the Maxwell equations in the context of the three approximations in page 26 above, we wish to make a non-relativistic/quasi-static approximation. Here, some degree of caution must be applied, as the notion of non-relativistic regime is not uniquely defined: there are several ways to take a non-relativistic limit of a set of equations, leading to very different limit equations. As an example, we may already see this with respect to the Lorentz transformations:

$$t' = \frac{t - xV/c^2}{\sqrt{1-(V/c)^2}}, \qquad x' = \frac{x - Vt}{\sqrt{1-(V/c)^2}},$$

which have two types of non-relativistic limit $|V|/c \ll 1$ (we refer to [63] and references therein for a more complete discussion). In the above, $V \in \mathbb{R}_+$ is a fixed velocity defining a change of inertial reference frames in the x direction. On the one hand, if ones assume the ultratimelike[12] condition $ct \gg |x|$, then one obtains the well-known Galilean transformations

$$t = t', \qquad x' = x - Vt,$$

whereas if the ultraspacelike[13] condition $ct \ll |x|$ is instead assumed, then we have the Caroll transformations[14] introduced by Lévy-Leblond [76],

$$t' = t - \frac{1}{c^2} V x, \qquad x = x'.$$

The existence of two different limits, which is linked to the presence of three different parameters (c, t, x), incites us to great care when dealing with the Maxwell equations. Many "non-relativistic approximations" of electromagnetism are stated in a very vague manner, and in fact do not yield Galilean invariant equations. If we want to attain a theory of magnetofluids that maximizes physical relevance, studying these is a necessary step.

In the context of Maxwell equations, there are, in addition to the speed of light c, two extra scalar quantities which are the electric permittivity ϵ_0 and magnetic permeability μ_0 of vacuum, which will define the scales for the electric and magnetic fields. Because in SI units these are linked to c by $\epsilon_0 \mu_0 c^2 = 1$, it is not possible to let $c \to +\infty$ alone in the equations while one of these quantities remain finite: a choice of balance must be made between ϵ_0 and μ_0. As a consequence, there are two different non-relativistic limits of electromagnetism: an *electric limit*

[12] We borrow the terminology from [63]. The condition $ct \gg |x|$ means that we restrict our attention to phenomena which have a limited spatial extension when compared to the duration of the experiment, so that the propagation of light and information within the system is nearly instantaneous.

[13] On the other hand, the ultraspacelike condition implies that the different elements of the system are so far apart that they cannot be causally connected in the duration of the experiment.

[14] They are named in reference to the author of *Alice in the Wonderland*, Lewis Carroll.

defined by $|e| \gg c|b|$ and a *magnetic limit* defined by $c|b| \gg |e|$. This fact was discovered by[15] Le Bellac and Lévy-Leblond [75].

We are interested in the magnetic limit $c|b| \gg |e|$ of the Maxwell equations, which is associated to the electromagnetic units $\mu_0 = 1$ and $\epsilon_0 = c^{-2}$ (see Section 3 in the Appendix of [65]). In that case, the Maxwell equations yield the following Galilean approximation:

$$\begin{array}{ll} \partial_t b = -\nabla \times e & \nabla \times b = j \\ \mathrm{div}(e) = \frac{1}{\epsilon_0}\varrho & \mathrm{div}(b) = 0. \end{array} \tag{14}$$

These equations are invariant under Galilean transformations, provided that the electric and magnetic fields transform as

$$e'(t, x + Vt) = e(t, x) + V \times b(t, x), \qquad b'(t, x) = b(t, x + Vt). \tag{15}$$

In particular, the magnetic field (and the electrical current) are Galilean invariant in the magnetic limit! These transformations for the electromagnetic field are in fact quite natural (see [63] and [91]) when considering how the Lorentz force $F = q(e + v \times b)$, acting on a particule of charge q and speed v, might be transformed by the Galilean group. Assume that F is, as a force, Galilean invariant. Then, by invariance of the electric charge,

$$F' = q\big(e' + v' \times b'\big) = q\big(e + (v + V) \times b\big) = F.$$

Because this quantity must hold for all particule velocities, we see that the electric field must transform as

$$e'(t, x + Vt) = e(t, x) + V \times b(t, x),$$

which is the part of the righthand side that does not depend linearly on v, and the magnetic field as $b'(t, x + Vt) = b(t, x)$. In some way, this means that the magnetic limit is "suited" to the Lorentz force. It should also be noted that the transformation of the electric field described above is very much linked to Faraday's law, and can be derived from it by Galilean considerations, see Section 5.15 in [65].

To conclude this paragraph, we recall a few notions concerning Ohm's law and conducting media. Given a conducting medium whose resistivity is $\mu(t, x)$, the local form of Ohm's law links the electric field and the electrical current vector and is written $e = \mu j$. In particular, we see that there is no electric field in a perfectly conducting medium $e = 0$. However, this is only true in an inertial reference in which the conductor is at rest, so that, in another reference frame in which the conductor has constant velocity V, the transformation laws (15) imply that the electric field is

$$e(t, x) = -V \times b(t, x). \tag{16}$$

This last equation is particularly interesting with regards to the magnetic limit (14) of the Maxwell equations, as in (14) there is only one evolution equation (on b) for two unknowns (b, e). The local Ohm's law (16) allows the unknown e for which we had no equation to be expressed in terms of the magnetic field and the medium's velocity.

Ideal MHD

In this paragraph, we use the discussion above on the Maxwell equations to derive the ideal MHD equations.[16] We will then explore some of the basic features of the PDE system.

[15] Most of what we say here is directly taken from [75].

[16] We refer to [11] for a derivation of MHD from the Vlasov equations. Although we stay at the hand-waving level for this formal introduction, rigorous mathematical work is also available. For example, the article [38] of Arsénio and Gallagher presents a derivation of MHD from a Navier-Stokes-Maxwell system.

Firstly, as we have explained, the fluid is assumed to be highly collisional, so that the kinetics can be described by means of a fluid equation. Because the fluid has constant density $\rho \equiv 1$ and is inviscid, we will work with the incompressible Euler equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \Pi = F \\ \operatorname{div}(u) = 0. \end{cases}$$

Here, F is the force the magnetic field exerts on the fluid. As the fluid is globally neutral, F cannot be expressed directly in terms of the charge density: we rather invoke the Laplace force $F = j \times b$, which is the resulting force of the Lorentz force on the charges whose motion creates the electrical current.

Next, we assume that the fluid is non-relativistic and dominated by its magnetic field, so that we may use the magnetic limit (14) of the Maxwell equations. In particular, the magnetic field is linked to the electrical current by $j = \nabla \times b$, so that the Laplace force reads

$$j \times b = (\nabla \times b) \times b = (b \cdot \nabla) b - \frac{1}{2} \nabla \left(|b|^2\right) = \operatorname{div}\left(b \otimes b - \frac{1}{2}|b|^2 \mathrm{Id}\right).$$

This alternate expression for the Laplace force provides a bit more visibility on how it acts on the fluid in terms of magnetic pressure and tension (Sections 9.3 to 9.5, pp. 268-272 in [11]). We will define the *magnetic pressure* Π_c as being the opposite of the trace of the magnetic stress tensor

$$\Pi_c = -\operatorname{Tr}\left(b \otimes b - \frac{1}{2}|b|^2 \mathrm{Id}\right) = \frac{1}{2}|b|^2,$$

and which acts on the fluid similarly to the usual pressure force.

Finally, the evolution of magnetic field is governed by the magnetic limit (14) of the Maxwell equations. The assumption that the fluid is perfectly conducting allows us to use Ohm's law applied to a medium in motion (16) and write

$$\partial_t b = -\nabla \times e = \nabla \times (u \times b) = (b \cdot \nabla) u - (u \cdot \nabla) b.$$

The equations on u and b are sufficient to describe the evolution of all the variables, and by putting them together we form the ideal MHD system:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \pi = (b \cdot \nabla) b \\ \partial_t b + (u \cdot \nabla) b = (b \cdot \nabla) u \\ \operatorname{div}(u) = 0, \end{cases} \tag{17}$$

where we have defined the MHD pressure π as being the sum of the usual hydrodynamic pressure and the magnetic pressure $\pi = \Pi + \Pi_c$. Recall that, in regards of our discussion of the Maxwell equations, there is no need to include the divergence constraint $\operatorname{div}(b) = 0$ in the system, as it is not needed for the dynamics of the system by virtue of (13). This property is directly inherited from the Maxwell equations.

One of the most striking aspects of our derivation of the ideal MHD equations, is that they were obtained from *Galilean* theories: Newton's laws in the form of the incompressible Euler system and the magnetic limit of the Maxwell equations. We therefore expect that the resulting PDEs (17) to share those Galilean invariance property. It turns out that this is the case: equations (17) are left unchanged by the transformations

$$u'(t, x + Vt) = u(t, x) + V, \qquad b'(t, x + Vt) = b(t, x), \qquad \pi'(t, x + Vt) = \pi(t, x).$$

This pleasant property shows that ideal MHD (17) is not a blunt set of approximations or an *ad hoc* system for plasma physics, but will have deep physical properties, such as conserved quantities. For example, it is possible to write an action principle for the MHD equations [17].

Conservation Laws and Elsässer Variables

The goal of this paragraph is to introduce a change of variables in the MHD system (17) which will be of the utmost importance for our work in the dissertation: the Elsässer variables. From a mathematical perspective, it is the possibility to recast the system in in these variables that enable us to solve it in the function spaces we need for our analysis. Here, we aim at showing how the Elsässer variables are not only a convenient mathematical tool, but are natural for the study of MHD, as they reveal the symmetry of the equations. We will illustrate this by giving some of the conservation laws for MHD.

Total Energy. By taking the scalar product of the momentum equation in (17) by u and the magnetic field equation by b, we obtain equations describing the balance of kinetic and magnetic[17] energy:

$$\begin{aligned} &\big(\partial_t + u \cdot \nabla\big) \tfrac{1}{2}|u|^2 + u \cdot \nabla\pi = (b \cdot \nabla) b \cdot u \\ &\big(\partial_t + u \cdot \nabla\big) \tfrac{1}{2}|b|^2 = (b \cdot \nabla) u \cdot b. \end{aligned}$$

Because the fluid is incompressible, the work of the pressure force does not affect the total kinetic energy, as $u \cdot \nabla\pi = \operatorname{div}(\pi u)$ is a total derivative, and so has a vanishing integral on the whole space. Similarly, both terms $(b \cdot \nabla) b \cdot u$ and $(b \cdot \nabla) u \cdot b$ are somewhat symmetric and their sum also is a total derivative

$$(b \cdot \nabla) b \cdot u + (b \cdot \nabla) u \cdot b = (b \cdot \nabla)(u \cdot b) = \operatorname{div}((u \cdot b) b),$$

so that they may be understood as energy flows between the fluid and the magnetic field. The overall energy conservation law reads

$$\big(\partial_t + u \cdot \nabla\big) E + \operatorname{div}(\pi u) = \operatorname{div}(H_c b),$$

where $E = \frac{1}{2}(|u|^2 + |b|^2)$ is the total energy density and $H_c := u \cdot b$. Since the fluid is incompressible, the transport of E by the flow of the divergence-free velocity field u implies that the integral $\int E \, \mathrm{d}x$ remains constant with respect to time.

Cross-Helicity. The quantity $H_c = u \cdot b$ appearing in the energy conservation law is also of interest: it is called the *cross-helicity* and also is a conserved quantity. Cross-helicity has an interpretation in terms of the topology of fluid and magnetic flux tubes for which physicists use it. Our main concern is that it is another conserved quantity that is *quadratic* with respect to the unknowns. By taking the scalar product of the momentum equation by b, of the magnetic field equation by u and summing, we obtain

$$\big(\partial_t + u \cdot \nabla\big) H_c + \operatorname{div}(\pi b) = \operatorname{div}(bE).$$

It should be noted that both conservation laws for the energy and cross-helicity are somewhat similar, in that the quantities are advected by the fluid velocity and affected by the MHD pressure and each other.

Remark. There are many more conserved quantities in ideal MHD, such as the magnetic helicity H_m which describes the topology of the magnetic flux tubes and is defined in terms of the magnetic[18] potential a by $H_m = u \cdot a$. In two dimensions,[19] an infinity of conserved quantities may be found, the norms $\|a\|_{L^p}$, for $1 \leq p \leq +\infty$, are independent of time.[20]

[17] The energy density in the magnetic field is $\frac{1}{2}|b|^2$. We refer to 5.16 pp. 212-215 in [65] for a discussion of this principle.

[18] This definition is not gauge invariant, but it turns out that the total magnetic helicity $\int H_m$ is. We refer to Section 10.6 in [11], pp. 320-322 for details.

[19] The meaning of 2D magnetohydrodynamics will be explained below.

[20] The results of [10] are worth mentioning: they construct low regularity solutions that violate the conservation laws. These ideas are linked to Taylor's conjecture concerning the approximate conservation of magnetic helicity in weakly resistive fluids and the related Woltjer-Taylor relaxation, see Chapter 11 in [11].

The apparent symmetry between the conservation laws for the energy and the cross-helicity hints at an underlying structure in the equations. This can be revealed by a simple change of variables: define the *Elsässer variables* by

$$\alpha = u + b \qquad \text{and} \qquad \beta = u - b.$$

They were introduced in 1950 by W. M. Elsässer [41] in order to study wave propagation in magnetofluids.[21] Their main advantage is that they provide equations that are relatively decoupled: by taking the sum and the difference of the first and second equations in (17), we get

$$\begin{cases} \partial_t \alpha + (\beta \cdot \nabla)\alpha + \nabla \pi = 0 \\ \partial_t \beta + (\alpha \cdot \nabla)\beta + \nabla \pi = 0 \\ \mathrm{div}(\alpha) = \mathrm{div}(\beta) = 0. \end{cases} \tag{18}$$

In particular, by taking the scalar product of the first equation with α and the second one with β, we get the conservation of *two* quantities

$$\|\alpha\|_{L^2} = \|\alpha_0\|_{L^2} \qquad \text{and} \qquad \|\beta\|_{L^2} = \|\beta_0\|_{L^2}.$$

The total energy and cross-helicity densities can be deduced from the Elsässer variables by $E = \frac{1}{4}|\alpha|^2 + \frac{1}{4}|\beta|^2$ and $H_c = \frac{1}{4}|\alpha|^2 - \frac{1}{4}|\beta|^2$. For this reason, E and H_c are sometimes referred to as the *Elsässer energies*. The fact that the square norms $|\alpha|^2$ and $|\beta|^2$ satisfy two decoupled conservation laws

$$\big(\partial_t + \beta \cdot \nabla\big)\tfrac{1}{2}|\alpha|^2 + \mathrm{div}(\pi\alpha) = 0$$
$$\big(\partial_t + \alpha \cdot \nabla\big)\tfrac{1}{2}|\beta|^2 + \mathrm{div}(\pi\beta) = 0$$

shows that (α, β) are a privileged set of variables for the system. In fact, as we will see, the existence of this structure is paramount to existence and uniqueness theory of solutions.

The reader may already have noted that the Elsässer system, as it stands in (18), is formally overdetermined, because the single incompressibility constraint $\mathrm{div}(u) = 0$ has been replaced by two divergence equations $\mathrm{div}(\alpha) = \mathrm{div}(\beta) = 0$. To restore a balance between the number of unknowns and of equations, and to avoid the awkward condition $\mathrm{div}(\alpha + \beta) = 0$, we introduce two *a priori different* pressure functions:[22]

$$\begin{cases} \partial_t \alpha + (\beta \cdot \nabla)\alpha + \nabla \pi_1 = 0 \\ \partial_t \beta + (\alpha \cdot \nabla)\beta + \nabla \pi_2 = 0 \\ \mathrm{div}(\alpha) = \mathrm{div}(\beta) = 0. \end{cases}$$

On the formal level, this creates no inconsistencies: the structure of the equations implies that, under appropriate conditions at $|x| \to +\infty$, we must have $\nabla\pi_1 = \nabla\pi_2 = \nabla\pi$, as both π_1 and π_2 are solutions of the same elliptic problem: taking the divergence of the first two equations give

$$-\Delta\pi_1 = \partial_j \beta_k \partial_k \alpha_j = \partial_j \alpha_k \partial_k \beta_j = -\Delta\pi_2.$$

As we will see below, this creates no real problem as long as we work with solutions in spaces that rule our harmonic functions, such as finite energy solutions. For more general infinite energy solutions, which play a central role in the dissertation, this issue is much more delicate.

[21] A striking feature of the Elsässer variables is that, if the magnetic field is constant $b = B_0 \in \mathbb{R}^d$, then α is exactly transported along the magnetic field lines $\partial_t\alpha - (B_0 \cdot \nabla)\alpha = 0$, while β is transported in the opposite direction. Therefore, a α and a β wave packet will not interfere unless they collide. See [57] for an application to turbulence of Alfvén waves.

[22] Schmidt in [95] has the same idea, but adds an extra variable in the magnetic field equation.

The Mathematics of MHD

We now turn to the more mathematical aspects of MHD, that is involving function spaces, operators and well-posedness of PDE systems. Our goal will be to explain how the theory of PDE solutions is especially difficult in ideal MHD and why specific tools are needed.

Incompressible Fluids

As we have pointed out above, dealing with incompressible fluids introduces a non-local behavior in the fluid. This is manifest in the fact that the pressure Π is an independent unknown in its own right due to the degeneracy of the state law.

In fact, the pressure in incompressible fluids can be entirely computed from the remaining unknowns. We take as an example the Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \Pi = 0 \\ \mathrm{div}(u) = 0. \end{cases} \tag{19}$$

By taking the divergence of the momentum equation, we get an elliptic equation solved by Π,

$$-\Delta \pi = \partial_j \partial_k (u_j u_k), \tag{20}$$

so that the pressure force can be expressed as $\nabla \pi = \nabla(-\Delta)^{-1} \mathrm{div}((u \cdot \nabla) u)$. Here, the inverted Laplace operator $(-\Delta)^{-1}$ can either be defined in terms of a convolution with the elementary solution of the Laplacian or by its Fourier transform. For instance, using the Fourier transform approach, we obtain

$$\widehat{\nabla \pi}(\xi) = \frac{\xi \otimes \xi}{|\xi|^2} . \mathcal{F}\big[(u \cdot \nabla) u\big](\xi). \tag{21}$$

Either way, the pressure is a non-local function of the velocity field. This creates inevitably difficulties in the theory of incompressible fluids as the regularity of non-local terms requires more sophisticated methods to be evaluated, such as pseudo-differential calculus, Littlewood-Paley analysis, *etc.*

On the other hand, (21) allows us to completely eliminate the pressure from the problem and reduce the system to a single equation depending on the velocity field alone. We defining, in accordance with (21), the *Leray projection operator*

$$\widehat{\mathbb{P} f}(\xi) = \left(\mathrm{Id} - \frac{\xi \otimes \xi}{|\xi|^2} \right) \widehat{f}(\xi), \tag{22}$$

we may replace the Euler system (19) by

$$\partial_t u + \mathbb{P} \, \mathrm{div}(u \otimes u) = 0. \tag{23}$$

Note that there is no further need for the incompressibility constraint $\mathrm{div}(u) = 0$ in this equation: because (20) is the divergence of the momentum equation, we see that the operator $\mathbb{P}$ is designed to have its range in the space of divergence-free functions. Is is therefore enough to enforce $\mathrm{div}(u) = 0$ at initial time $t = 0$ for the solution of (23) to remain incompressible at all subsequent times. We can go one step further: $\mathbb{P}$ is the orthogonal projection (in the sense of L^2) on the space of divergence-free functions, $\mathrm{Id} - \mathbb{P}$ being the orthogonal projector on the space of functions f that are the gradients of some function $f = \nabla g$.

Operators of the form (22) are called *Fourier multipliers*, and are of special interest in the study of incompressible hydrodynamics. More precisely, for any function $\varphi(\xi)$ of the Fourier variable, the Fourier multiplier of symbol $\varphi(\xi)$ is the operator $\varphi(D)$ defined by the Fourier transform

$$\forall f \in \mathcal{S}, \qquad \widehat{\varphi(D) f}(\xi) = \varphi(\xi) \widehat{f}(\xi).$$

Because of the inherent non-locality of incompressible fluids, a number of important operators assume this form. We give two additional examples. Firstly, we should mention the Biot-Savart law, which allows a divergence-free $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ to be recovered from the curl matrix $[\mathrm{curl}\,(f)]_{ij} = \partial_j f_i - \partial_i f_j$ through the formula

$$f_k = (-\Delta)^{-1} \sum_{j=1}^{d} \partial_j \big[\mathrm{curl}\,(f)\big]_{jk}. \tag{24}$$

The interest of studying the curl matrices is that they often are solutions of simple equations: for example, the vorticity matrix $\omega = \mathrm{curl}\,(u)$ associated to the flow of the incompressible Euler system (19) is solution of a transport-like equation obtained by taking the curl of the first equation in (19), namely

$$\partial_t \omega + (u \cdot \nabla)\omega = {}^t(\nabla u)^2 - (\nabla u)^2.$$

In particular, this operation has the advantage of completely eliminating the pressure from the problem. Another related example worth mentioning is the link between the deformation tensor $\nabla_\sigma u$ and the flow. For a divergence-free f as above, define the symmetric matrix $[\nabla_\sigma f] = \frac{1}{2}(\partial_i f_j + \partial_j f_i)$. Then the field f can be obtained by

$$f = 2(-\Delta)^{-1}\,\mathrm{div}(\nabla_\sigma f).$$

As the Leray projector and the Biot-Savart law, we see that f is the image of $\nabla_\sigma f$ by a Fourier multiplier.[23] We refer to Section 4.6 for more on the equations solved by the deformation tensor $\nabla_\sigma u$ of an incompressible flow.

The omnipresence of Fourier multiplication operators in incompressible fluid mechanics creates serious difficulties. The most obvious one is that of an appropriate functional framework. We illustrate this with the Leray projection operator (22), whose symbol $m(\xi)$ is a bounded homogeneous function of degree zero. The Plancherel theorem immediately provides $L^2 \longrightarrow L^2$ continuity of $\mathbb{P}$ as

$$\|\mathbb{P}f\|_{L^2}^2 = \int |m(\xi)\widehat{f}(\xi)|^2 \mathrm{d}\xi \leq C\|\widehat{f}\|_{L^2}^2 \approx \|\, f\,\|_{L^2}^2.$$

However, $L^p \longrightarrow L^p$ boundedness is much more difficult to achieve (by means of Calderón-Zygmund theory if $1 < p < +\infty$, see Section 1.6). In fact it is not even true that the operator $\mathbb{P}$ is bounded on L^1 or L^∞ (see Section 1.2). We will therefore have to resort to substitute spaces for L^∞, such as Besov spaces or spaces of functions of bounded mean oscillations. Besov spaces especially will play a predominant role in our work: they are *ad hoc* spaces tailored to Fourier multipliers.

Remark. Before moving on, we must introduce a *nuance* to what we have said immediately above: the reader may have the impression that incompressibility can only be a source of problems and complex behavior (non-locality, non-boundedness of key operators on usual spaces, *etc.*), but this is not the entire picture. The pressure (or equivalently, the Leray projector) also has a beneficial effect on the equations. For example, if the pressure is removed from the Euler equations along with the associated incompressibility contraint, then the resulting equation is a Burgers-type equation and does not possess global regular solutions for generic initial data (see Chapter 4 in [42]). This contrasts with the existence of global regular solutions for the 2D Euler problem: the presence of the pressure seems to avert the blowup of solutions.

2D Well-Posedness Theory

In this paragraph, we aim at explaining the specific difficulties in the theory of solutions of the ideal MHD equations. While ideal MHD may seem at first glance to be a simple generalization of

[23] This notion is of course linked to the Korn inequality, see Remark 4.14.

the Euler equations, there is a striking difference: even in the simpler case of a plane fluid $d = 2$, *it is still unknown whether the ideal MHD equations possess global unique solutions.* In fact, this has been a notoriously challenging problem in the topic of mathematical magnetohydrodynamics for the past forty years since the initial work of Alekseev [2] in 1982.

In what follows below, we will explore the source of this difference by investigating the well-posedness issue for increasingly more difficult variants of the 2D ideal MHD system: we will consider the MHD system with additional dissipation terms

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla\pi = (b \cdot \nabla)b + \nu\Delta u \\ \partial_t b + (u \cdot \nabla)b = (b \cdot \nabla)u + \mu\Delta b \\ \mathrm{div}(u) = 0, \end{cases} \tag{25}$$

where $\nu, \mu \geq 0$ are respectively the viscosity and resistivity of the fluid, and we will progressively set the coefficients μ and ν to zero in order to evaluate the difficulties that will appear. We point out that our summary introduction on the matter may be complemented with the survey [103].

Before we delve in the rest of the discussion, we must clarify what is precisely meant by plane MHD, or a 2D magnetofluid, as the magnetic field has an intrinsically three-dimensional nature: the Biot-Savart law implies that the field lines circle around the electrical currents.

From a strictly algebraic point of view, the MHD equations (25) can be written in any[24] dimension $d \geq 2$, as the equations do not contain any operation specific to the 3D setting, such as vector cross-products, curl operators, *etc.* But solutions of the 2D system also have an interpretation in terms of the full 3D one: any solution (u, b, Π) of (25) with $d = 2$ defines a solution (u', b', Π') of the 3D system by setting, for $x = (x_1, x_2, x_3) \in \mathbb{R}^d$,

$$\begin{aligned} u'(t,x) &= (u_1(t,x_1,x_2), u_2(t,x_1,x_2), 0), \\ b'(t,x) &= (b_1(t,x_1,x_2), b_2(t,x_1,x_2), 0), \\ \Pi'(t,x) &= \Pi(t,x_1,x_2). \end{aligned}$$

In other words, the solution is invariant under translation along the vertical axis. In this configuration, the electrical current is always directed by the third vector $e_3 = (0,0,1)$ of the basis and crosses normally the plane of the fluid, and can therefore be represented by a scalar function $j = \partial_1 b_2 - \partial_2 b_1$. The induced magnetic field circles around the current, and therefore only has x_1 and x_2 components. The same remarks apply to the vorticity ω and the plane velocity field u.

Finally, we must point out that, as for the Euler equations, the context of two dimensions also has a number of specific features. For example, it is possible to construct exact stationary solutions with compact support (see Subsection 4.6.1 for more), while a virial argument due to Shafranov [98] shows that there can be no such solution for three-dimensional MHD. Another interesting property of plane MHD is the existence of an infinity of conserved quantities: because the magnetic field has no divergence, it is the derivative of a magnetic potential $b = \nabla^\perp a$ which solves a pure transport equation

$$\partial_t a + u \cdot \nabla a = 0,$$

so all the L^p norms of a are conserved $\|a\|_{L^p} = \|a_0\|_{L^p}$. This is to be compared to the conservation of the L^p norms of the scalar vorticity ω in the 2D Euler equations.

The case $\nu, \mu > 0$

We begin our study of system (25) with the case of the fully dissipative MHD, where both the viscosity and resistivity are positive $\nu, \mu > 0$. In this case, which is by far the easiest regarding

[24]The incompressibility constraint $\mathrm{div}(u) = 0$ implies that incompressible fluids on the line $d = 1$ are of limited interest.

the well-posedness problem, the MHD equations are totally parabolic: both evolution equations in (25) are heat equations up to non-linear terms of smaller order.

As with the Navier-Stokes equations, the natural approach is to perform energy estimates to find the natural class of functions to construct solutions in. By multiplying the first equation in (25) by the velocity u, we obtain a kinetic energy balance equation:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\,\mathrm{d}x + \nu\int |\nabla u|^2\,\mathrm{d}x = \int (b\cdot\nabla)b\cdot u\,\mathrm{d}x. \tag{26}$$

Similarly, by multiplying the magnetic field equation by b, we get a magnetic energy balance equation, which reads

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b|^2\,\mathrm{d}x + \mu\int |\nabla b|^2\,\mathrm{d}x = \int (u\cdot\nabla)b\cdot u\,\mathrm{d}x. \tag{27}$$

Now, integration by parts and the fact that both u and b are divergence-free show that both righthand side parts of (26) and (27) are opposite. This means that the kinetic energy that the fluid gains because of the Laplace force is precisely the energy the magnetic field looses because of its interaction with the fluid, as we have shown with local conservation laws above. We therefore have a total energy conservation law[25]

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \Big\{|u|^2+|b|^2\Big\}\,\mathrm{d}x + \int \Big\{\nu|\nabla u|^2+\mu|\nabla b|^2\Big\}\,\mathrm{d}x = 0. \tag{28}$$

The natural space of functions in which to seek solutions is therefore, as with the Navier-Stokes equations, the Sobolev space $u, b \in L^\infty(L^2)\cap L^2(\dot H^1)$. And indeed, as with the Navier-Stokes equations, we get existence of Leray-type solutions which are also unique.

Theorem. *Let $u_0, b_0 \in L^2(\mathbb{R}^2)$ be two divergence-free functions. There exists a unique global weak solution $(u,b)\in L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^2))\cap L^2(\mathbb{R}_+;\dot H^1(\mathbb{R}^2))$ associated to these initial data.*

In fact, the proof of this theorem is nearly identical to its Navier-Stokes counterpart, since the parabolic MHD system (25) (with $\nu,\mu>0$) is a Generalized-Navier-Stokes system. We refer the the fifth chapter of [7] for a discussion of these systems.

The case $\nu = 0, \mu > 0$

We now add some difficulty and assume that the fluid is inviscid $\nu = 0$. In this case, the basic energy estimates (28) above provide

$$u\in L^\infty(L^2)\qquad\text{and}\qquad b\in L^\infty(L^2)\cap L^2(\dot H^1), \tag{29}$$

which are insufficient to construct solutions, let alone prove their uniqueness, because of the lack of compactness regarding the velocity field. In order to overcome this, we must find higher order estimates. With that in mind, we define the vorticity and electrical current by

$$\omega = \partial_1 u_2 - \partial_2 u_1 \qquad\text{and}\qquad j = \partial_1 b_2 - \partial_2 b_1.$$

We will note $\mathrm{curl}\,(f) = \partial_1 f_2 - \partial_2 f_1$ the curl of a plane vector field. The fact that both u and b have no divergence provides two crucial properties of ω and j.

[25] Total energy is in fact decreasing. As is usual in mathematics, we are quite loose in terminology with "conservation laws", as what really matters is the boundedness of key functional norms.

1. Firstly, the vector fields u and b can be entirely recovered from ω and j, thanks to the Biot-Savart law (24), which can be written
$$u = \nabla^\perp(-\Delta)^{-1}\omega, \qquad \text{where } \nabla^\perp = (-\partial_2, \partial_1),$$
and likewise for b. By taking the Fourier transform of the previous formula, we immediately see that[26] ω and j do indeed provide first order estimates on u and b. Namely,
$$\|\nabla b\|_{L^2} \leq \|j\|_{L^2} \qquad \text{and} \qquad \|\nabla^2 b\|_{L^2} \leq \|\nabla j\|_{L^2}. \tag{30}$$

2. Secondly, if $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are two divergence-free vector fields, the curl of $(f\cdot\nabla)g$ takes the following form:
$$\begin{aligned} \operatorname{curl}(f\cdot\nabla)g &= f\cdot\nabla\operatorname{curl}(g) + \partial_1 f_1\Big(\partial_1 g_2 + \partial_2 g_1\Big) - \partial_1 g_1\Big(\partial_1 f_2 + \partial_2 f_1\Big) \\ &= f\cdot\nabla\operatorname{curl}(g) + \mathcal{L}(\nabla f, \nabla g). \end{aligned} \tag{31}$$
In particular, if $f = g$, then the bilinear term $\mathcal{L}(\nabla f, \nabla f)$, which is in fact skew-symmetric, cancels and $\operatorname{curl}(f\cdot\nabla)f = f\cdot\nabla f$. This algebraic miracle is very specific to plane incompressible fluids, and is the main reason for the global well-posedness of the Euler equations.

As a consequence of the second point, we find a system of two scalar PDEs for ω and j by applying the curl operator to the equations and using (31). We get the equations
$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = b\cdot\nabla j \\ \partial_t j + u\cdot\nabla j - b\cdot\nabla\omega - \mu\Delta j = 2\mathcal{L}(\nabla u, \nabla b), \end{cases}$$
on which we may perform energy estimates. By testing the first equation with ω and the second one with j, we obtain
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\Big\{\omega^2 + j^2\Big\}\,\mathrm{d}x + \mu\int|\nabla j|^2\,\mathrm{d}x = \int\mathcal{L}(\nabla u, \nabla b)j\,\mathrm{d}x + \int\Big\{\omega b\cdot\nabla j - jb\cdot\nabla\omega\Big\}\,\mathrm{d}x. \tag{32}$$
Integration by parts shows that this last integral is zero, so that we only need to estimate the first one on the righthand side to obtain a differential inequality. By using the Hölder and Gagliardo-Nirenberg inequalities[27], we get
$$\left|\int\mathcal{L}(\nabla u, \nabla b)j\,\mathrm{d}x\right| \leq \|\nabla u\|_{L^2}\|\nabla b\|^2_{L^4} \leq C\|\nabla u\|_{L^2}\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}.$$
Using the inequalities of (30), as well as the Young inequality $ab \leq \frac{1}{2}\mu a^2 + 2\mu^{-1}b^2$, we get
$$\left|\int\mathcal{L}(\nabla u, \nabla b)j\,\mathrm{d}x\right| \leq C(\mu)\|\nabla b\|^2_{L^2}\|\omega\|^2_{L^2} + \frac{1}{2}\mu\|\nabla j\|^2_{L^2}, \tag{33}$$
which, combined with (32) gives the differential inequality we were looking for:
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\Big\{\omega^2 + j^2\Big\}\,\mathrm{d}x + \frac{1}{2}\mu\int|\nabla j|^2\,\mathrm{d}x \leq C(\mu)\|\nabla b\|^2_{L^2}\|\omega\|^2_{L^2}.$$
Because of the basic energy conservation law (28), the function $M(t) := \|\nabla b(t)\|^2_{L^2}$ is globally integrable $M \in L^1(\mathbb{R}^+)$. Therefore, applying Grönwall's lemma to the differential inequality (33) and combining with (29) provides global bounds for (u, b) in the following space:
$$u \in L^\infty(H^1) \qquad \text{and} \qquad b \in L^\infty(H^1)\cap L^2(\dot{H}^2),$$
and these are sufficient to construct weak solutions to (25) with $\nu = 0$.

[26]In fact, the inequalities below are equalities, thanks to the fact that u is divergence-free. This is again seen on the Fourier transform, as $i\xi\cdot\widehat{u}(\xi) = 0$.

[27]The Gagliardo-Nirenberg inequality states that, for all $p \in [2, +\infty[$, and $f \in H^1(\mathbb{R}^2)$, we have $\|f\|_{L^p} \leq C(p)\|f\|^{2/p}_{L^2}\|\nabla f\|^{1-2/p}_{L^2}$. See for example Corollary 1.2 in [21].

Theorem. (Kozono, 1989, [70]) *Assume $\mu > 0$ and $\nu = 0$. Let $u_0, b_0 \in H^1$ be two divergence-free functions. There exists a weak solution $(u, b) \in L^\infty(H^1) \times L^\infty(H^1) \cap L^2(H^2)$ to problem* (25).

Remark. The careful reader will have noticed that the solutions in the previous Theorem are *weak*. This means that some of the terms in system (25) may be only defined in the sense of distributions, and not as locally integrable functions. The problem lies with the pressure force $-\nabla\pi$, which is given by Leray projection

$$\nabla\pi = \nabla(-\Delta)^{-1} \sum_{k,j} \partial_k \partial_j (u_k u_j - b_k b_j) = \nabla(-\Delta)^{-1} \operatorname{div} \Big((u \cdot \nabla) u - (b \cdot \nabla) b \Big).$$

Whatever way we study it, proper bounds seem out of reach. For instance, since $u \in L^\infty(H^1)$, the product $u \otimes u$ lies[28] in every $L^\infty(H^{1-\epsilon})$ for $\epsilon > 0$, and so $\nabla\pi \in L^\infty(H^{-\epsilon})$. Similarly, the functions $(u \cdot \nabla)u$ is in $L^\infty(L^1)$, but the operator $\nabla(-\Delta)^{-1} \operatorname{div}$ fails to map L^1 into even the space $\mathcal{M}$ of finite measures! (see Section 1.2)

Although global weak solutions exists for our problem with $\mu > 0$ and $\nu = 0$, their uniqueness remains to be proved. Concerning this problem, we refer to the helpful survey [103], which covers this problem through a number of interesting angles.

The case $\nu > 0, \mu = 0$

From the previous discussion, we already may expect the case $\mu = 0, \nu > 0$ to be much more difficult. The reason is that when taking the curl of the system

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega - \Delta \omega = b \cdot \nabla j \\ \partial_t j + u \cdot \nabla j - b \cdot \nabla \omega = \mathcal{L}(\nabla u, \nabla b), \end{cases}$$

and performing energy estimates, we obtain an inequality which is analogous to (32),

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \Big\{ \omega^2 + j^2 \Big\} \, \mathrm{d}x + \nu \int |\nabla \omega|^2 \, \mathrm{d}x \leq \left| \int \mathcal{L}(\nabla u, \nabla b) j \, \mathrm{d}x \right|. \tag{34}$$

Now, bounding the righthand side is the hard part: it is quadratic with respect to derivatives of b, whereas the lefthand side can only provide a H^1 bound on b. Let us elaborate. A natural inequality to write is the simple Hölder bound

$$\left| \int \mathcal{L}(\nabla u, \nabla b) j \, \mathrm{d}x \right| \leq \|\nabla u\|_{L^\infty} \|j\|_{L^2} \|\nabla b\|_{L^2} \leq \|\nabla u\|_{L^\infty} \|j\|_{L^2}^2.$$

This inequality is, in a sense, optimal, since we have taken full advantage of the H^1 bound the lefthand side of (34) will allow. To close the estimates, we would need an L^∞ bound on ∇u, but only have at our disposal an $H^1 \not\subset L^\infty$ one, making it impossible to find global *a priori* estimates.

It is possible to find *a priori* estimates in spaces of functions for which u will be Lipschitz, however, these will fail to be global with respect to time. In fact, global existence of solutions is still unknown, even in the case of small initial data. The only step in that direction is a global result near a well-chosen equilibrium [80].

Concerning local well-posedness, the best result that has been obtained so far is the following, for initial data in homogeneous Besov spaces[29] (which holds in all dimensions $d \geq 2$).

[28] In two dimensions, the space H^1 is a *critical* Sobolev space in the sense that H^s is an algebra if and only if $s > 1$. This is closely linked with Sobolev embeddings: $H^s \hookrightarrow L^\infty$ if and only if $s > 1$ (given $d = 2$).

[29] Besov spaces will be defined in the first Chapter of the dissertation. Here, we simply mention that, for regularity and integrability exponents $s \in \mathbb{R}$ and $p \in [1, +\infty]$, the Besov space $\dot{B}^s_{p,1}$ is slightly smaller than the corresponding Sobolev space $\dot{W}^{s,p}$.

Theorem. (Li, Tan, Yin, 2017, [79]) *Assume $\mu = 0$ and $\nu > 0$ and consider divergence-free initial data u_0 and b_0 which lie in the homogeneous Besov spaces*

$$u_0 \in \dot{B}^{-1+d/p}_{p,1} \qquad \textit{and} \qquad b_0 \in \dot{B}^{d/p}_{p,1}, \qquad \text{for } 1 \leq p \leq 2d. \tag{35}$$

Then there is a time $T > 0$ such that problem (25) has a unique solution (u, b) in the space

$$(u, b) \in C^0([0, T[; \dot{B}^{-1+d/p}_{p,1}) \cap L^1([0, T[; \dot{B}^{1+d/p}_{p,1}) \times C^0([0, T[; \dot{B}^{d/p}_{p,1}).$$

This theorem is the last (for the time being!) of a number of statements, the first of which being the work of Fefferman, McCormick, Robinson and Rodrigo [47] for initial data $u_0, b_0 \in H^s$, with $s > d/2$, followed by Chemin, and McCormick, Robinson and Rodrigo [22], who dealt with initial data in optimal Besov spaces (that is (35) with $p = 2$). The last work before [79] was again by Fefferman, McCormick, Robinson and Rodrigo [48], in nearly optimal Sobolev spaces $(u, b) \in H^{s-1-\epsilon} \times H^s$, for $s > d/2$. We refer to the discussion in [103] for other references and further comments.

The case $\nu = \mu = 0$

Our work is mainly concerned with ideal MHD: $\mu = \nu = 0$. From the explanations above, it is obvious that finding global solutions in that case will be immensely challenging. As a matter of fact, the problem has resisted the continued investigations of mathematicians for the best part of forty years.

As with the non-resistive case $\nu > 0$ and $\mu = 0$, it is possible to construct local solutions of the ideal MHD system. The functional setting appropriate for these solutions may be guessed by observing, as above, the vorticity equations. The estimate (34), with $\nu = 0$, can be closed only by imposing that two of the quantities ∇b, j and ∇u are bounded. A natural guess would be to require the solutions to be in $u, b \in H^s$ with $s > 2$ so that $H^s \subset W^{1,\infty}$. We will extensively comment on this in the next Section.

Summary of the Dissertation

In this final Section, we layout the main ideas developed in our work and present our results. Although we have provided a lengthy introduction to the general topic of MHD, this Summary is meant to be relatively independent.

Statement of the Problem

The bulk of this dissertation concerns the existence and uniqueness issue for the initial value problem of the ideal MHD equations. These are a system of PDEs, which read

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \pi = (b \cdot \nabla) b \\ \partial_t b + (u \cdot \nabla) b = (b \cdot \nabla) u \\ \mathrm{div}(u) = 0, \end{cases} \tag{36}$$

We will study these equations in the whole space $\mathbb{R}^d$. This choice, as we will see, will have a significant impact on the methods we use and the nature of our results: it is not only a comfortable setting giving us access to an array of harmonic analysis techniques (working on the torus $\mathbb{T}^d$ would do just as well). The mere fact of considering incompressible fluids which may have infinite spatial extension will have deep consequences.

In the equations (36), $u(t, x) \in \mathbb{R}^d$ represents the (average) velocity vector of the fluid particles at point $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}$, while $b(t, x) \in \mathbb{R}^d$ is the value of the magnetic field. The scalar function $\pi(t, x)$ is the MHD pressure, a combination of the usual hydrodynamic pressure and the magnetic pressure (see system (17) above).

As we have said before, well-posedness of (36) is a delicate problem. A simple glance at the equations is convincing: both equations for u and b are transport equations, and will have propagation of regularity properties. However, the righthand side terms imply derivatives of the unknown, so they are less regular than u and b themselves, and we might expect that solutions instantly degenerate, jeopardizing even the possibility for solutions to exist on a small time scale.

It so happens that this pessimistic scenario is, in reality, avoided, due to the simple fact that (36) has a fundamental physical property: conservation of energy. The total kinetic and magnetic energy is constant in time

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \left\{ |u|^2 + |b|^2 \right\} \mathrm{d}x = 0.$$

This confers the equations a special structure: it is a quasi-linear symmetric hyperbolic system. As such, local existence of solutions in supercritical[30] Sobolev spaces H^s with $s > 1 + d/2$.

Theorem. (Schmidt, 1987, [95]) *Let $u_0, b_0 \in H^s$ be divergence free initial data with $s > 1 + d/2$. Then there exists a $T > 0$ and a unique solution $(u, b) \in C^0([0, T[; H^s)$ of* (36) *associated to the initial data (u_0, b_0).*

However, the method we have described is limited to L^2-based spaces, as it uses energy techniques in a very fundamental way. This should not be surprising, given that symmetric hyperbolic systems are known to be ill-posed in L^p for all other values $p \neq 2$ (see [15]). This is quite unfortunate, as finding solutions in L^p-based spaces for $p > 2$ has many advantages, the main one being the possible regularity of solutions.

Given the form of the ideal MHD equations (36), we are looking for solutions in spaces contained in that of Lipschitz functions $W^{1,\infty}$. In terms of embeddings (Sobolev embeddings for example), we are looking for solutions whose regularity exponent is $s \geq 1 + d/p$. Therefore, taking p as large as possible allows us to lower the regularity exponent. To see the full advantage gained from this, consider the endpoint exponent $p = +\infty$, so that $s = 1$ and limit the discussion to the simpler 2D Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \Pi = 0 \\ \operatorname{div}(u) = 0. \end{cases} \tag{37}$$

These are only the ideal MHD equations (36) in which we have set $b = 0$, and where the full MHD pressure π is limited to the hydrodynamic pressure Π. Then, because we work in two dimensions of space $d = 2$, the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ solves a pure transport equation

$$\partial_t \omega + u \cdot \nabla \omega = 0. \tag{38}$$

Ordinarily, if the flow u has regularity $s = 1 + d/p$, the vorticity ω would have regularity $s - 1 = d/p$, and standard theory for the transport equation would provide estimates for ω of the form

$$\|\omega\|_{L^\infty_T(X^{s-1})} \leq \|\omega_0\|_{X^{s-1}} \exp\left(C \int_0^T \|\nabla u\|_{L^\infty} \, \mathrm{d}t \right), \tag{39}$$

where X^{s-1} is a space of functions of regularity $s - 1 = d/p$. This is in sharp contrast with the simple conservation of norms $\|\omega\|_{L^p} = \|\omega_0\|_{L^p}$ one obtains in the zero regularity spaces L^p due to incompressibility. To get closer to this pleasant conservation of norms, we want to push all the

[30] We agree that the word *supercritical* the following meaning: an exponent is supercritical if it is *higher* than the critical value, here $1 + d/2$.

way to the endpoint exponent $p=+\infty$, so that $X^{s-1}=X^0$ is a space of functions with $s-1=0$ regularity, in other words, something very close to the Lebesgue space L^∞.

We may mention several well-posedness results for the 2D Euler equations that are in this spirit. For example, proving the existence and uniqueness of the celebrated Yudovich solutions (see Chapter 8 in [81]) uses extensively the conservation of $L^1\cap L^\infty$ norms of ω, while the Serfati solutions [97] (see also [3]) crucially depends on the fact that $\|\omega\|_{L^\infty}$ is constant. Lastly, the proof of Hmidi and Keraani [64] will be the most important for us, as it is the only one that can be suitably generalized to MHD. The authors of [64], instead of constructing solutions with $\omega\in L^\infty$, consider the slightly smaller Besov space $X^0 := B^0_{\infty,1}\subset L^\infty$. As this space behaves closely to L^∞, having regularity exponent $s=1=0$, the exponential inequality (39) can be replaced by the bound

$$\|\omega\|_{B^0_{\infty,1}} \lesssim \|\omega_0\|_{B^0_{\infty,1}} \left(1+\int_0^T \|\nabla u\|_{L^\infty}\,\mathrm{d}t\right), \tag{40}$$

which is linear with respect to the transport field u, and so yields a global estimate thanks to Grönwall's lemma.

The Role of Elsässer Variables

Let us return to the ideal MHD system (36). So far, we have fancifully described the advantages of constructing solutions in L^p-based spaces with $p\neq 2$, but general hyperbolic theory is not powerful enough to depart from $p=2$. We must resort to deeper properties of our system. These are provided by a simple change of variables: the Elsässer variables, introduced in 1950 by the German-American physicist W.M. Elsässer,[31]

$$\alpha = u+b \qquad \text{and} \qquad \beta = u-b.$$

As we have shown above, the quantities α and β decouple in a way some of the conserved quantities of MHD (the so-called Elsässer energies), and in fact they allow the problem to be written as a system of transport equations, up to two non-local pressure terms:

$$\begin{cases} \partial_t\alpha + (\beta\cdot\nabla)\alpha + \nabla\pi_1 = 0\\ \partial_t\beta + (\alpha\cdot\nabla)\beta + \nabla\pi_2 = 0\\ \mathrm{div}(\alpha)=\mathrm{div}(\beta)=0. \end{cases}$$

In the equations above, π_1 and π_2 are (not necessarily equal) pressure functions associated to the two *independent* divergence equations $\mathrm{div}(\alpha)=0$ and $\mathrm{div}(\beta)=0$. The more symmetric aspect of the equations allows them to possess unique solutions in spaces based on L^p for $p\neq 2$. This is the spirit of the work of Secchi [96], who constructs solutions in Sobolev spaces $W^{m,p}$ with $m>1+d/p$ and $1<p<+\infty$. Unfortunately, the endpoint exponent $p=+\infty$, and therefore solutions (α,β) of regularity $s=1$ remain out of reach at that point.

The main obstacle keeping from reaching $p=+\infty$ is that the Leray projection operator $\mathbb{P}$ (see (21) above for a definition) is not a $L^\infty\longrightarrow L^\infty$ bounded operator: although it is a Singular Integral Operator, the usual Calderón-Zygmund theory fails to provide boundedness in L^∞. In addition, the solutions constructed by Secchi must have supercritical regularity $m>1+d/p$, so that achieving $s=1$ regularity will also require more advanced techniques that cover critical regularity $s=1+d/p$ solutions.

[31] In the physical literature, they are often noted $z^\pm = u\pm b$, while there is no clear consensus of notation among mathematicians.

The Role of Besov Spaces

The answer to both of these problems is found by adopting a functional framework better suited to the Leray projector $\mathbb{P}$, that of Besov spaces. These are based on the observation that, while the operator $\mathbb{P}$ is ill-defined in the $L^\infty \longrightarrow L^\infty$ topology, the restricted operator T_ϕ defined by

$$\forall f \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}^d), \qquad \widehat{T_\phi f}(\xi) = \left(1 - \frac{\xi \otimes \xi}{|\xi|^2}\right) \phi(\xi) \widehat{f}(\xi) \tag{41}$$

is $L^p \longrightarrow L^p$ bounded for the full range of exponents $p \in [1, +\infty]$, provided that $\phi \in \mathcal{D}$ be supported away from both $\xi = 0$ and $|\xi| \to +\infty$, point at which the symbol of $\mathbb{P}$ is discontinuous. On this ground, we define a dyadic partition of unity: let $\varphi(\xi)$ be a smooth and radial function supported in an annulus such that

$$1 = \sum_{j \in \mathbb{Z}} \varphi(2^j \xi) := \sum_{j \in \mathbb{Z}} \varphi_j(\xi), \qquad \text{for all } \xi \neq 0. \tag{42}$$

Then, for all f, we define the operator $\dot{\Delta}_j$ by $\mathcal{F}[\dot{\Delta}_j f](\xi) = \varphi_j(\xi) \widehat{f}(\xi)$. Formally, any function can be reconstructed from the $\dot{\Delta}_k f$ by using the sum

$$f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f.$$

This is called the *Littlewood-Paley decomposition* of the function f. Because of what we have said concerning the operators T_ϕ in (41) above, we know that the operators $\dot{\Delta}_j \mathbb{P}$ are continuous on all Lebesgue spaces L^p for $p \in [1, +\infty]$. By extension, the Leray projection operator $\mathbb{P}$ is bounded on the space $\dot{B}^s_p$ defined by the (semi) norm

$$\|f\|_{\dot{B}^s_p} := \sum_{j \in \mathbb{Z}} \|(-\Delta)^{s/2} \dot{\Delta}_j f\|_{L^p}. \tag{43}$$

The space $\dot{B}^s_p$ is a first example of a homogeneous Besov space (in fact $\dot{B}^s_p = \dot{B}^s_{p,1}$) and a glimpse at the methods which are the backbone of our present work.

Having provided a rough description of our problem and the methods we will use, we attempt to give an overview of our results. We have chosen to present these in the order they appear in the dissertation below, chapter by chapter.

We will start by Chapter 1, which contains the functional analysis material we will use in the rest of the dissertation, principally around homogeneous and non-homogeneous Besov spaces. There are also a few elements of original research concerning the role of Chemin's space $\mathcal{S}'_h$ of homogeneous distributions in the structure of homogeneous Besov spaces. Chapter 2 presents notions that are specific to the $p = +\infty$ setting of bounded solutions. In particular, it is concerned with minimal far-field conditions bounded solutions must satisfy to be determined by initial data. In Chapter 3, we finally study local well-posedness of the ideal MHD equations and prove a number of statements concerning the lifespan of solutions, mainly continuation criteria and lower bounds. The purpose of Chapter 4 is to study the vorticity form of ideal MHD. We will prove that the major results of Chapter 3 can also be established by using these alternative techniques, and we will comment on the algebra of the equations. Finally, in Chapter 5, we will show that our ideal MHD system (36) (with an added density perturbation function) can be seen as a limit of a fast-rotating ideal MHD system with non-constant density by means of a relative entropy inequality.

Chapter 1. Study of Besov Spaces

The importance of the Littlewood-Paley decomposition and Besov spaces is such that the whole of Chapter 1 will be dedicated to them. In addition to giving details about how they are an inescapable part of our analysis and providing definitions and basic properties (completeness, isomorphisms and embeddings, relation to paradifferential calculus, *etc.*), we will spend a significant amount of time discussing homogeneous Besov spaces and how they relate to usual realizations.

More precisely, we will be interested in Chemin's space $\mathcal{S}'_h$ of homogeneous distributions and its role in the structure of supercritical homogeneous Besov spaces.[32] A tempered distribution belongs to $\mathcal{S}'_h$ if and only if it satisfies a low frequency condition, namely

$$\chi(\lambda\xi)\widehat{f}(\xi) \underset{\lambda\to+\infty}{\longrightarrow} 0 \qquad \text{as } \lambda \to +\infty, \tag{44}$$

where $\chi \in \mathcal{D}$ is a fixed cut-off function such that $\chi(\xi) = 1$ for $|\xi| \leq 1$. Belonging to $\mathcal{S}'_h$ can also be seen as a form of far-field condition: a function $f \in \mathcal{S}'_h$ is expected to have some (very weak) kind of cancellation at infinity $|x| \to +\infty$. For example, any periodic function with average value zero is in $\mathcal{S}'_h$. The main purpose of the space $\mathcal{S}'_h$, and the reason it was defined by Chemin, is to serve as a basis for realizations of homogeneous Besov spaces.

The issue is that the space $\|\,.\,\|_{\dot{B}^s_p}$, as defined in (43) is only in fact a *semi-norm*, as $\|f\|_{\dot{B}^s_p} = 0$ as soon as the Fourier transform $\widehat{f}(\xi)$ is supported at $\xi = 0$, *i.e.* as soon as $f \in \mathbb{R}[X]$ is a polynomial function. This is of course linked to the fact that the Littlewood-Paley decomposition (42) only holds for $\xi \neq 0$. There are two possible courses of action to solve this problem. The first one is simply to work modulo polynomials: this is the option that is usually preferred in the functional analysis literature. The space $\dot{B}^s_p$ is then defined as the set of $f \in \mathcal{S}'/\,\mathbb{R}[X]$ such that $\|f\|_{\dot{B}^s_p}$ is finite. On the other hand, working modulo polynomials is unpleasant, to say the least, when dealing with nonlinear PDEs. This is why it was proposed to use instead the space

$$\dot{\mathfrak{B}}^s_p := \left\{ f \in \mathcal{S}'_h, \quad \|f\|_{\dot{B}^s_p} < +\infty \right\}$$

as a substitute for $\dot{B}^s_p$. Because of condition (44), there can be no polynomial function in $\dot{\mathfrak{B}}^s_p$, and $\|\,.\,\|_{\dot{B}^s_p}$ therefore defines a true norm.

Regretfully, significant differences exist between these two approaches. While the space $\dot{B}^s_p$ defined as a subspace of $\mathcal{S}'/\,\mathbb{R}[X]$ is always complete, it is not so for $\dot{\mathfrak{B}}^s_p$, which is not a Banach space if the exponent is supercritical $s > d/p$. Although all this is common knowledge, very little has been said on exactly how different the two strategies actually are. More generally, we dedicate Section 1.7 to the study between the differences between these two spaces. Our main results take the following form (see Proposition 1.37 and Theorems 1.40 and 1.41).

Theorem 1. *Let $s \in \mathbb{R}$ and $p, r \in [1, +\infty]$ such that the Besov space $\dot{B}^s_{p,r}$ (defined as a subspace of $\mathcal{S}'/\,\mathbb{R}[X]$) is supercritical: $s > d/p$, or $s = d/p$ and $r > 1$. Consider the space $\dot{\mathfrak{B}}^s_{p,r}$ of distributions $f \in \mathcal{S}'_h$ such that $\|f\|_{\dot{B}^s_{p,r}}$ is finite. Then*

(i) *The space $\dot{\mathfrak{B}}^s_{p,r}$ is not closed in $\dot{B}^s_{p,r}$ and is dense if and only if $r = +\infty$;*

(ii) *If $r = +\infty$, then the we may define C^s_p as the closure of $\dot{\mathfrak{B}}^s_{p,\infty}$ in the $\dot{B}^s_{p,\infty}$ topology. The quotient $\dot{B}^s_{p,\infty}/C^s_p$ is not separable.*

(iii) *The space C^s_p is uncomplemented in $\dot{B}^s_{p,\infty}$. In other words, there is no decomposition $\dot{B}^s_{p,\infty} = C^s_p \oplus G$ with continuous projections.*

[32] A homogeneous Besov space $\dot{B}^s_{p,r}$ will be called supercritical if its regularity exponent satisfies $s > d/p$, or $s = d/p$ and $r > 1$.

The various notions in Theorem 1 show how radically incompatible both approaches of homogeneous spaces end up in the supercritical case: for example, in point *(ii)*, the non separability of the quotient $\dot{B}^s_{p,\infty}/C^s_p$ must be understood as a measure of the difference between the two spaces, while the non-complementation property of point *(iii)* is a direct prolongation of a stronger property: the existence of a quasi-isometrical[33] embedding $\ell^\infty/c_0 \hookrightarrow \dot{B}^s_{p,\infty}/C^s_p$.The method we use in *(iii)* is a generalization of an argument of Whitley [102] in his proof of the Phillips-Sobczyk theorem stating that c_0 is uncomplemented in ℓ^∞, see [90], [99].

Chapter 2. Bounded Solutions

As we have explained above, constructing solutions in L^p-based spaces with p as large as possible is of manifold interest. But this is not without problem, as the case $p = +\infty$ poses many challenges that do not exist for $1 < p < +\infty$, especially from a functional analysis point of view. In addition, there is another source of difficulty linked to the general framework of solutions that lie in spaces of bounded functions: in themselves, they are not uniquely determined by the initial datum, even under strong smoothness conditions.

To explain further, consider the case of the incompressible Euler equations (37) and define a solution by

$$u(t,x) = f(t) \qquad \text{and} \qquad \Pi(t,x) = -f'(t)\cdot x, \tag{45}$$

where $f \in C^\infty(\mathbb{R};\mathbb{R}^d)$ is a smooth function. In particular, taking $f \neq 0$ such that $f(0) = 0$, we may construct an infinity of bounded solutions associated to the initial value $u_0 = 0$. This surprising breakdown of uniqueness can be understood in a couple of ways. Firstly, we may see the flow as being driven by a pressure differential applied at infinity, and of course, a flow under the action of an arbitrary "external" forcing cannot hope to be deterministic without prior knowledge of the forcing. Equivalently, the pressure differential $\nabla\Pi(t,x) = f'(t)$ can also be seen as an inertial force related to a change of non-inertial reference frames: for any solution u of the Euler equations, define a new solution of the Euler equations by

$$u'(t,x+F(t)) = u(t,x) + f(t) \qquad \text{and} \qquad \Pi'(t,x+F(t)) = \Pi(t,x) - f'(t)\cdot x, \tag{46}$$

where F is a primitive of f. This *generalized Galilean transform* corresponds simply to viewing the solution u in the accelerated reference frame given by the change of coordinates $x' = x + F(t)$. If f were constant, then we would simply find the Galileo transformations.

This question is intimately connected to the possibility of using the Leray projection operator for bounded solutions of the Euler equations. On the one hand, it is possible to write the projected equations

$$\partial_t u + \mathbb{P}\,\mathrm{div}(u\otimes u) = 0 \tag{47}$$

and make sense of them for bounded solutions: the Leray projection is not well defined as a $L^\infty \longrightarrow L^\infty$ operator, but the order one operator $\mathbb{P}\,\mathrm{div} : L^\infty \longrightarrow B^{-1}_{\infty,\infty}$ is. By using this fact, Pak and Park [87] construct Besov-Lipschitz solutions to problem (47) which are *uniquely determined* by the initial datum $u_0 \in B^1_{\infty,1}$. On the other hand, while all solutions of (47) are solutions of the Euler equations, the reverse cannot be true, otherwise the uniqueness of [87] would also hold for the original Euler equations.

In order to make sense out of all of this, we ask the following question: *On what condition is a solution of the Euler equations a solution of the projected problem* (47)*?*

[33] Recall that c_0 is the Banach space of sequences that converge to zero.

Many different attempts have been made to provide answers to this question, mainly in the form of sufficient conditions on the solutions. For example, if the flow remains bounded $u(t) \in L^\infty$ and the pressure satisfies[34]

$$\Pi(t) \in \mathrm{BMO} \qquad \text{or} \qquad \Pi(t,x) \underset{|x|\to+\infty}{=} o(|x|),$$

then u is automatically a solution of the projected equation (47), see respectively the articles of J. Kato [67] and Kukavica-Vicol [72] for instance. Other conditions may bear on the velocity field, as in the book of Lemarié-Rieusset [77] (Theorem 11.1), the paper of Lemarié-Rieusset and Fernández-Dalgo [54] or our own work [29]. We refer to Chapter 2 for more references on the topic.

In Chapter 2, we give a full presentation and proof of our main result on this question, Theorem 2.2, which we reproduce here below. As it is stated, it represents an improved version of the one contained in our paper [25].

Theorem 2. *Let $T > 0$, $u_0 \in L^\infty$ and $u \in C^0([0,T[;L^\infty)$ be a weak solution of the Euler equations (37) associated to the initial datum u_0 and to a pressure $\pi \in \mathcal{D}'([0,T[\times\mathbb{R}^d)$. Then the following assertions are equivalent:*

(i) *the flow u solves the projected problem with initial datum $u(0)$ that satisfies $u(0) - u_0 \in \mathrm{Cst} \in \mathbb{R}$,*

(ii) *for all times $t \in [0,T[$, we have $u(t) - u(0) \in \mathcal{S}'_h$,*

(iii) *for all times $t \in [0,T[$, we have $u(t) - u(0) \in \mathrm{BMO}^{-1}$,*

(iv) *the pressure satisfies $\pi \in C^0(]0,T[;\mathrm{BMO})$,*

(v) *the pressure force is continuous with respect to time $\nabla\pi \in C^0(]0,T[;\mathcal{S}')$ and $\nabla\pi(t) \in \mathcal{S}'_h$ for all $0 < t < T$.*

In some way, Theorem 2 is optimal, as it provides necessary and sufficient condition for solutions of the Euler equations (37) to solve the projected problem, both in terms of the pressure or the velocity field. It thus strictly improves some of the previous results while showing that some others are sharp, such as [67] for example.

In terms of method and proof, the first step in understanding the problem is to to prove that any bounded solution u of the Euler equations can be deduced from a solution of the projected problem (47) from a generalized Galilean transformation, as in (46). In other words, there exists a function of time $g : \mathbb{R} \longrightarrow \mathbb{R}^d$ such that

$$\partial_t u + \mathbb{P}\,\mathrm{div}(u \otimes u) + g(t) = 0,$$

so that u is a solution of (47) if and only if $g \equiv 0$. Because at each time $g(t)$ is a constant function of x, deciding whether $g(t) = 0$ or not can be done by looking only at low frequencies: the Fourier transform of $g(t)$ is a multiple of the Dirac mass $\delta_0(\xi)$. Therefore, we take a low-frequency cut-off $\chi(\xi)$, which is smooth, compactly supported and has value $\chi(\xi) = 1$ for $|\xi| \leq 1$ and apply the Fourier multiplier $\chi(\lambda D)$ to the previous equation after a scaling. We want to study the $\lambda \to +\infty$ asymptotic of

$$\partial_t \chi(\lambda D)u + \chi(\lambda D)\,\mathrm{div}(u \otimes u) + g(t) = 0.$$

[34] For the time being, we state these conditions without precise assumptions on u or Π. The reader may assume C^∞ smoothness of all functions without loosing the essence of the arguments: as is witnessed by (45), the issue does not lie in the smoothness of the solutions.

Now, and this is the core of our proof, the second summand in this equation tends to zero as $\lambda \to +\infty$. More precisely, we have

$$\big\|\chi(\lambda D)\mathbb{P}\operatorname{div}(u\otimes u)\big\|_{L^\infty} = O\left(\frac{1}{\lambda}\log(\lambda)\right) \qquad \text{as } \lambda \to +\infty.$$

A very intuitive way of understanding this estimate would be to resort to the first Bernstein inequality (see Lemma 1.7). Because the symbol of $\mathbb{P}\operatorname{div}$ is of order one, a $O(\lambda^{-1})$ decay is expected. Unfortunately, the symbol of $\mathbb{P}\operatorname{div}$ is not regular enough for the Bernstein inequalities to apply (its first derivatives are not continuous), so we must use finer integral estimates to obtain the sought decay. In the end, the last step is to note that, at the limit $\lambda \to +\infty$, we see that $g(t) = 0$ if and only if $\partial_t\chi(\lambda D)u \longrightarrow 0$, or in other words $\partial_t u \in \mathcal{S}'_h$. The condition $u(t) - u(0) \in \mathcal{S}'_h$ of Theorem 2 is an integrated version of this condition.

The detailed proof in Chapter 2 covers solutions with very rough regularity, which require careful approximation techniques and special attention to the functional framework used. The techniques that we develop are also applied to Serfati solutions: we prove that a bounded solution of the Euler system satisfying the Serfati identity is actually a solution of (47).

Chapter 3. Elsässer Variables and Well-Posedness

Chapter 3 is concerned with the application of all the techniques we described above to the ideal MHD system. As such, it is in some way the heart of the dissertation. We will study the initial value problem for the ideal MHD system recast in Elsässer variables:

$$\begin{cases} \partial_t\alpha + (\beta\cdot\nabla)\alpha + \nabla\pi_1 = 0\\ \partial_t\beta + (\alpha\cdot\nabla)\beta + \nabla\pi_2 = 0\\ \operatorname{div}(\alpha) = \operatorname{div}(\beta) = 0. \end{cases} \tag{48}$$

Obviously, bounded solutions of (48) will present the same problems we have just discussed for the Euler equations: solutions are not unique unless some kind of condition be imposed on the pressure or the velocity field. But there is a further, related though slightly different, problem we must face: although the Elsässer system (48) was obtained from the ideal MHD equations (36) by a simple change of variables, both systems are not, as they stand equivalent. Even if any solution (u, b) of the original MHD system (36) defines a solution $(\alpha, \beta) = (u + b, u - b)$ of (48), the converse is not true: taking the sum and the difference of both equations for α and β yields the magnetic field equation *but with an additional gradient term*:

$$\partial_t b + \operatorname{div}(u\otimes b - b\otimes u) = \frac{1}{2}\nabla(\pi_2 - \pi_1),$$

so that (u, b) is a solution of the original MHD system only if $\nabla\pi_1 = \nabla\pi_2$. Once again, a simple example will suffice to show that this may happen: consider

$$\alpha(t,x) = f(t) = -\beta(t,x), \qquad \pi_1(t,x) = -f'(t)\cdot x = \pi_2(t,x).$$

Then $(u, b) = \frac{1}{2}(\alpha+\beta, \alpha-\beta)$ is not a solution of (36) as soon as $f'(t) \neq 0$.

A series of computations much like those of Chapter 2, though quite simpler, will find a necessary and sufficient condition the *magnetic field* must fulfill in order for a solution of (48) to solve ideal MHD (36), namely $b(t) - b(0) \in \mathcal{S}'_h$. This is the content of Theorem 3.11 that is reproduced here.

Theorem 3. *Let $T > 0$, $\alpha_0, \beta_0 \in L^\infty$ and $(\alpha, \beta) \in C^0\big([0, T[; L^\infty\big)$. Define $(u, b) = \frac{1}{2}(\alpha + \beta, \alpha - \beta)$, and (u_0, b_0) accordingly. The assertions below are true.*

1. *Assume that (u, b) is a weak solution of (3.1) that is related to the initial data (u_0, b_0). Then (α, β) solves the Elsässer system (48) with initial data (α_0, β_0) and $\pi_1 = \pi_2 = \pi$.*

2. *Assume that (α, β) is a weak solution of (48) for the initial data (α_0, β_0). Then the following statements are equivalent:*

   (i) *the functions (u, b) solve the "classical" MHD system (36) with initial data $(u_0, b(0))$, and the difference $b(0) - b_0$ is a constant function;*

   (ii) *we have, for all times $t \in [0, T[$, $b(t) - b(0) \in \mathcal{S}'_h$.*

   In addition, if one of these equivalent conditions is fulfilled, then the pressure gradients $\nabla \pi_1$ and $\nabla \pi_2$ are equal.

In addition, by generalizing the techniques deployed in Chapter 2, we may apply the Leray projection operator to the momentum equation, so as to eliminate the MHD pressure, provided that the condition $u(t) - u(0) \in \mathcal{S}'_h$ holds on the velocity field. In that case, we obtain the following equivalent systems

$$
\begin{cases} \partial_t u + \mathbb{P}\,\mathrm{div}(u \otimes u - b \otimes b) = 0 \\ \partial_t b + \mathrm{div}(u \otimes b - b \otimes u) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} \partial_t \alpha + \mathbb{P}\,\mathrm{div}(\beta \otimes \alpha) = 0 \\ \partial_t \beta + \mathbb{P}\,\mathrm{div}(\alpha \otimes \beta) = 0. \end{cases} \tag{49}
$$

The relations between all the different systems we have considered may be summarized in the following diagram.

$$
\begin{tikzcd}
\mathrm{Els} \arrow[r, "\alpha, \beta"] \arrow[d, "b"] & \mathbb{P}(\mathrm{Els}) \arrow[l, bend left] \arrow[d, shift right] \\
\mathrm{MHD} \arrow[r, "u"] \arrow[u, bend left] & \mathbb{P}(\mathrm{MHD}) \arrow[l, bend left] \arrow[u, shift right]
\end{tikzcd}
$$

In the above, Els stands for the Elsässer system (48) while MHD stands for the classical MHD system (36). The names $\mathbb{P}$(MHD) and $\mathbb{P}$(Els) refer to, respectively, the projected equations in (49). The arrows are labeled with the conditions required to pass from a system to another, bare arrows need no conditions. For example, the label u stands for the condition $u(t) - u(0) \in \mathcal{S}'_h$.

The main point of the preceding discussion is that we have managed to recast, under some mild far field conditions, ideal MHD in a system of transport equations, up to commutator terms, namely

$$
\begin{cases} \partial_t \alpha + (\beta \cdot \nabla)\alpha = \big[\beta \cdot \nabla, \mathbb{P}\big]\alpha \\ \partial_t \beta + (\alpha \cdot \nabla)\beta = \big[\alpha \cdot \nabla, \mathbb{P}\big]\beta, \end{cases}
$$

on which we may apply the standard Littlewood-Paley machinery: in order to get *a priori* estimates for the solutions in Besov spaces, we must find L^p estimates on the functions $\Delta_j f$ (see (43) above), which are found by applying the operator Δ_j to the preceding transport equations. For example, doing to in the first equation yields

$$
\Big(\partial_t + \beta \cdot \nabla\Big)\Delta_j \alpha = \Delta_j \big[\beta \cdot \nabla, \mathbb{P}\big]\alpha + \big[\beta \cdot \nabla, \Delta_j\big]\alpha.
$$

Estimating the commutators in the righthand side is a familiar procedure that has been used many times in the last decades (see the classical textbook [7]). It is based on the Bony paraproduct decomposition and elements of paradifferential calculus. Overall, provided we work in a Besov space of Lipschitz functions $B^s_{p,r} \subset W^{1,\infty}$, or in other words $s \geq 1 + d/p$ with $r = 1$ in the case of equality $s = 1 + d/p$, we get a quadratic inequality on α,

$$\|\alpha\|_{L^\infty(B^s_{p,r})} \lesssim \|\alpha_0\|_{B^s_{p,r}} + \int_0^T \|\alpha\|_{B^s_{p,r}} \|\beta\|_{B^s_{p,r}} \, \mathrm{d}t,$$

which, with the analogous inequality for β, provides local well-posedness for the equations on a finite time interval $[0, T[$, and the lifespan T^* of the solution can be bounded from below by the inequality

$$T^* \geq \frac{C}{\|(\alpha, \beta)\|_{B^s_{p,r}}},$$

which is of the same form than the lower bounds obtained from hyperbolic theory, albeit in more general spaces. The local existence and uniqueness of solutions is the result of Theorem 3.12, restated here.

Theorem 4. *Let $p, r \in [1, +\infty]$ and $s \in \mathbb{R}$ such that the Besov space $B^s_{p,r}$ is supercritical: $B^s_{p,r} \subset W^{1,\infty}$ or in other words*

$$s > 1 + \frac{d}{p} \qquad \text{or} \qquad s = 1 + \frac{d}{p} \text{ and } r = 1. \tag{50}$$

Consider divergence-free initial data $u_0, b_0 \in B^s_{p,r}$. There is a time $T > 0$ such that the system (3.1) has a unique solution $(u, b) \in C^0([0, T[; B^s_{p,r})$ if $r < +\infty$, and[35] in $C^0_w([0, T[; B^s_{p,r})$ if $r = +\infty$, associated to the initial data (u_0, b_0) which satisfies, if $p = +\infty$,

$$u(t) - u(0) \in \mathcal{S}'_h.$$

In addition, if $r < +\infty$, we also have $u, b \in C^1([0, T[; B^{s-1}_{p,r})$ and $\partial_t u, \partial_t b \in C^0_w([0, T[; B^{s-1}_{p,r})$ when $r = +\infty$. Finally, the time T can be chosen so that

$$T \geq \frac{C}{\|u_0, b_0\|_{B^s_{p,r}}}, \tag{51}$$

for some constant $C = C(d, s, p, r)$.

Existence and uniqueness of solutions of ideal MHD in Besov spaces is not in itself new: it has been proved by Miao and Yuan [83]. However, our own result [28], [29], [25] brings two contributions to the topic. Firstly, the earlier proof of [83] rests on compositions of solutions by the flows of $u \pm b$, and is therefore highly technical. By directly using Elsässer variables, we propose a massive simplification. Furthermore, we bring additional clarification as to the meaning of uniqueness in the $p = +\infty$ framework of bounded (Besov-Lipschitz) solutions which was not present before, previous results only dealing with the projected equations (49) instead of the the original, more general, MHD equations featuring the pressure Π as an unknown.

Local well-posedness of (36) is only the first step in the study of the lifespan T^* of solutions. In a second part of Chapter 3, we focus on a number of continuation criteria which use and highlight the specific features of the equations. The first one, and most simple, of these goes as follows: the unique solution $(u, b) \in C^0([0, T[; B^s_{p,r})$ obtained in Theorem 4 can be continued beyond time T if and only if (see Proposition 3.22)

$$\int_0^T \Big\{ \|\Delta_{-1}(u, b)\|_{L^\infty} + \|\nabla (u, b)\|_{L^\infty} \Big\} \, \mathrm{d}t < +\infty. \tag{52}$$

[35]The symbol $C^0_w([0, T[; X)$ refers to measurable functions $f : [0, T[\longrightarrow X$ with values in a Banach space with a predual Y who are continuous for the weak topology: for all $\phi \in Y$, the bracket $t \mapsto \langle f(t), \phi \rangle_{X \times Y}$ is continuous on $[0, T[$.

Here, Δ_{-1} is a low frequency cut-off: the function $\Delta_{-1}f$ depends only on values of $\widehat{f}(\xi)$ for $|\xi| \leq 1$. This result, of the same kind than the well-known Beale-Kato-Majda criterion [8], is to be compared with other similar ones for ideal MHD: that of Caflish, Klapper and Steele [16] bearing on $\|(\omega, j)\|_{L^\infty}$, or the those of subsequent papers [104], [18] or [23] (see Chapter 3) which improve the regularity of the norms in the integral. The novelty of (52) is rather that it covers the case of bounded solutions $p = \infty$. In fact, the presence of the low frequency term $\|\Delta_{-1}(u, b)\|_{L^\infty}$ is due to this particularity, and completely disappears if the initial data possess some kind of integrability, say $\alpha_0, \beta_0 \in L^q$ for some $q < +\infty$ (Corollary 3.23). Note that the time requirements on the low frequency part of the solution may be improved through a kind of logarithmic interpolation: Proposition 3.24 shows that (52) can be replaced by

$$\int_0^T \|(\nabla u, \nabla b)\|_{L^\infty} \left\{ 1 + \log\left(1 + \frac{\|\Delta_{-1}(u, b)\|_{L^\infty}}{\|(\nabla u, \nabla b)\|_{L^\infty}}\right)\right\} \mathrm{d}t < +\infty.$$

Other continuation criteria we find take advantage of the structure of the equations. For example, the magnetic field equation in (36) is *linear*[36] with respect to b. It is therefore expected that b should not blow-up unless u also does. This translates in a continuation criterion expressed in terms of u alone, Theorem 3.25. Similarly, we may note that the each equation in the Elsässer system (48) is also linear with respect to α or β, so that blow-up can also be detected on α or β alone: this is the origin of the criteria of Theorem 3.27. These results were published in our article [30].

In the last part of Chapter 3, we investigate the lifespan of solutions in the regime of low magnetic fields. Our motivation comes from the two dimensional setting, where the equations are known to globally well-posed if $b \equiv 0$. Assuming the lifespan has some sort of continuity property with respect to the initial data, we would expect the T^* to become arbitrarily large as the initial magnetic field b_0 becomes small in two dimensions $d = 2$.

Such a property is absent from the lower bound (51) that is already at our disposal: taking $b_0 \longrightarrow 0$ only leads to a finite lower bound. In Theorem 3.29, we use the Elsässer variables to compare MHD solutions with a regular "target" solution of the Euler equations and obtain, in the case of two dimensions $d = 2$, a quantitative inequality on the lifespan,

$$T \geq \frac{C}{\|u_0\|_{L^2 \cap B^2_{\infty,1}}} \log\left\{1 + C \log\left[1 + C \log\left(1 + C\frac{\|u_0\|_{L^2 \cap B^2_{\infty,1}}}{\|b_0\|_{B^1_{\infty,1}}}\right)\right]\right\}. \tag{53}$$

This inequality is an improvement, published in [30] of one of our previous results [29] which we will present below in our summary of Chapter 4, as the methods used are quite different.

Chapter 4. Vorticity Form of the Elsässer System

In Chapter 4, we focus on the vorticity form of the Elsässer system (48). The goal is to express the problem in terms of the order one variables $\omega \pm j$. Strictly speaking, little is proven that has not already been in Chapter 3, but the arguments and methods are quite different, which is why we include them in the dissertation. We hope that the multiplicity of different techniques may contribute to our understanding of the problem.

With Chapter 4, we come back to the theme we developed above: we have explained that one of the interests of constructing solutions in L^p-based spaces with p as large as possible is to work with a regularity exponent which is as small as possible. Because the solutions constructed in Theorem 4 have $s = 1 + d/p$ regularity, the endpoint exponent $p = +\infty$ provides a setting where we have two quantities of regularity $s - 1 = 0$, namely

$$X = \operatorname{curl}(\alpha) = \omega + j \qquad \text{and} \qquad Y = \operatorname{curl}(\beta) = \omega - j.$$

[36] Many thanks to Raphaël Danchin for this remark!

This level of regularity is particularly pleasant, because of the availability of improved estimates for the transport equation, such as the linear inequality (40). We therefore seek to determine the form of the equations solved by X and Y. Taking the curl of the Elsässer system (48), we find

$$\begin{cases} (\partial_t + \beta \cdot \nabla)X = \mathcal{L}(\nabla\alpha, \nabla\beta) \\ (\partial_t + \alpha \cdot \nabla)Y = \mathcal{L}(\nabla\beta, \nabla\alpha) \end{cases}$$

where $\mathcal{L}(\nabla\alpha, \nabla\beta)$ is a matrix valued bilinear combination of the coefficients of $\nabla\alpha$ and $\nabla\beta$ given by

$$\big[\mathcal{L}(\nabla\alpha, \nabla\beta)\big]_{ij} = \partial_j\beta_k\partial_k\alpha_i - \partial_i\beta_k\partial_k\alpha_j.$$

As we have said, our goal is to find order zero estimates for X and Y, these quantities being in the Besov space $B^0_{\infty,1}$. But since X and Y are solutions of transport equations, with the bilinear $\mathcal{L}$ considered as forcing terms, we may apply the linear inequality (40) and get

$$\begin{aligned} (54)\quad \|(X,Y)\|_{L^\infty_T(B^0_{\infty,1})} \lesssim &\left(1 + \int_0^T \|\nabla(\alpha,\beta)\|_{L^\infty}\,\mathrm{d}t\right) \\ &\times \left\{\|(X_0,Y_0)\|_{B^0_{\infty,1}} + \int_0^T \|\mathcal{L}(\nabla\alpha,\nabla\beta)\|_{B^0_{\infty,1}}\right\}. \end{aligned}$$

Let us now focus on the two dimensional case $d = 2$, with a principle in mind: staying as close as possible to the *a priori* estimates we have described above for the 2D Euler equations. We would like to quantify the difference between inequality (54) just above and the linear inequality (40) obtained when there is no magnetic field so as to obtain a lower bound for the lifespan of solutions which increases as b_0 is small, in the spirit of (53). We must therefore study the bilinear term $\mathcal{L}(\nabla\alpha, \nabla\beta)$ in order to make the magnetic field appear.

This is made possible thanks to a bit of algebra specific to the 2D setting. Because $\mathcal{L}(\nabla\alpha, \nabla\beta)$ must be zero if $\alpha = \beta$, in which case the MHD equations reduce to the Euler equations and the vorticity equations to the pure transport equation (38), the bilinear map $\mathcal{L}$ must be skew-symmetric. Expressing α and β as functions of the physical variables, we obtain

$$\mathcal{L}(\nabla\alpha, \nabla\beta) = \mathcal{L}(\nabla(u+b), \nabla(u-b)) = -2\mathcal{L}(\nabla u, \nabla b),$$

and we are left with estimating the norm $\|\mathcal{L}(\nabla u, \nabla b)\|_{B^0_{\infty,1}}$. This can be done with a bit of work: the space $B^0_{\infty,1}$ is not a Banach algebra, so it is not only a matter of straightforward computation, but requires additional care, see Lemma 4.5. But in the end, we have

$$\|(X,Y)\|_{L^\infty_T(B^0_{\infty,1})} \lesssim \left(1 + \int_0^T \|\nabla(\alpha,\beta)\|_{L^\infty}\,\mathrm{d}t\right)\left\{\|(X_0,Y_0)\|_{B^0_{\infty,1}} + \int_0^T \|u\|_{B^1_{\infty,1}}\|b\|_{B^1_{\infty,1}}\right\}.$$

This inequality is close to a linear one in the regime of low magnetic fields. This means that carefully isolating the dependency on the initial magnetic field b_0 while performing estimates, we should obtain a lower bound for the lifespan of solutions which is a function of b_0. In the end, we have an inequality (Theorem 4.10) that is similar to (53),

$$T \geq \frac{C}{\|(u_0,b_0)\|_{B^2_{\infty,1}\cap L^2}} \log\left\{1 + \log\left[1 + \log\left(1 + C\frac{\|(u_0,b_0)\|_{B^1_{\infty,1}\cap L^2}}{\|b_0\|_{B^1_{\infty,1}}}\right)\right]\right\},$$

but less precise because it requires both initial fields to have order two regularity $u_0, b_0 \in B^2_{\infty,1}$. This should be seen as a disadvantage of the method we have used: in all the argument we have developed here, the magnetic field was used as a measure of the difference between the ideal MHD system and the Euler equations. But to estimate $\|b\|_{B^1_{\infty,1}}$ in terms of the initial magnetic field alone, we need to use the magnetic field equation

$$\partial_t b + (u \cdot \nabla) b = (b \cdot \nabla) u,$$

which involves a higher order derivative of u and explains why we need a full $B^2_{\infty,1}$ solution at the basis of the computations. The proof of inequality (53) was much different, as other better-suited quantities were used to measure the difference between ideal MHD and the Euler equations.

As a last part of this paragraph, we mention a few, yet unpublished, computations which conclude Chapter 4. These are concerned with further exploring the algebraic structure of plane MHD by finding equations on the symmetric gradients

$$\big[\nabla_\sigma \alpha\big]_{ij} = \frac{1}{2}\Big(\partial_i \alpha_j + \partial_j \alpha_i\Big)$$

and $\nabla_\sigma \beta$. After the derivation of a new system, we use it to study linear solutions

$$\alpha(t,x) = A(t).x \qquad \text{and} \qquad \beta(t,x) = B(t).x,$$

where $A(t), B(t) \in C^1(\mathbb{R}^d \otimes \mathbb{R}^d)$ are time dependent matrices. Such linear solutions, already present in [81], were studied for the 3D Euler and Navier-Stokes equations in [84] to determine their blow-up properties. In contrast, writing the differential equations solved by $A(t)$ and $B(t)$, we find they are actually *linear*, and so have global solutions.

Chapter 5. Fast Rotating MHD

In the final Chapter, we study a fast-rotation asymptotic for an incompressible density-dependent MHD problem, thus continuing our previous treatment[37] of the problem [27]. The goal is to determine the dynamics of geophysical fluids, which are characterized the Coriolis force which dominates their dynamics. The material in Chapter 5 was published in our article [28].

The physical setting for this problem is that of large scale fluids lying on a rotating celestial body, *e.g.* oceanic or atmospheric flows, magma or stellar fluids. The importance of the rotation and its effects through Coriolis force is described by a dimensionless parameter, the Rossby number Ro, whose size is related to the balance between the kinetics of the fluid and the Coriolis force. In very broad terms, the Rossby number is inversely proportional to the rotation speed of the reference frame.

The primitive equations we begin with will be those of the 2D incompressible MHD system with a variable density $\rho(t,x) \geq 0$, which read[38]

$$\begin{cases} \rho\, \partial_t u + \rho (u \cdot \nabla) u + \frac{1}{\epsilon} \nabla \Pi + \frac{1}{\epsilon} \rho u^\perp = h(\epsilon) \operatorname{div}(\nu(\rho) \nabla u) + (b \cdot \nabla) b - \frac{1}{2} \nabla(|b|^2) \\ \partial_t b + (u \cdot \nabla) b = (b \cdot \nabla) u + \operatorname{curl}(\mu(\rho) \operatorname{curl}(b)) \\ \partial_t \rho + u \cdot \nabla \rho = 0 \\ \operatorname{div}(u) = 0. \end{cases} \tag{55}$$

The Rossby number is noted $Ro = \epsilon$, thus obliquely introducing our goal, studying the limit dynamics $\epsilon \to 0^+$. Here, the scalar quantities $h(\epsilon)\nu(\rho)$ and $h(\epsilon)\mu(\rho)$ are the viscosity and resistivity of the fluid, which we allow to depend on the density ρ and scale as $h(\epsilon) > 0$ and $\epsilon^{-1} \rho u^\perp$ is the Coriolis force.

[37] The results of [27] have been the object of a masters thesis and will not be reproduced here.

[38] In this system, the magnetic field equation is directly derived from the local Ohm law $e = -u \times b = \mu(\rho) j$ in the reference frame where the fluid is motionless. Using the Maxwell-Faraday equation yields the desired PDE.

The two dimensional and incompressible setting might be frowned at at first, but there are multiple reasons that dictate this choice. The first one is entirely technical: a fast rotation limit of (55) in a full three dimensional setting is way beyond our reach and presents, at this date, immense difficulties. Secondly, there also are physical justifications to our choice, so our problem is not rendered irrelevant. It is a characteristic feature of geophysical fluids that they are nearly incompressible and have a 2D dynamics. This principle, known as the Taylor-Proudman theorem, has been rigorously proved in many different settings. For example, we mention the work of Feireisl, Gallagher and Novotný [50] concerning compressible fluids and the textbook [21] for more on the topic.

In our previous work [27], we have studied the $\epsilon \to 0^+$ asymptotics of (55) in the case where $h(\epsilon) = 1$ and with *ill-prepared* initial data: the initial values of (ρ, u, b) were aloud to depend on ϵ as long as

$$(b_{0,\epsilon})_{\epsilon>0} \subset L^2, \qquad (m_{0,\epsilon})_{\epsilon>0} \subset L^2, \qquad \left(\frac{|m_{0,\epsilon}|^2}{\rho_{0,\epsilon}}\right)_{\epsilon>0} \subset L^1, \qquad (\rho_{0,\epsilon})_{\epsilon>0} \subset L^\infty$$

where the notation $(f_\epsilon)_\epsilon \subset X$ means that the sequence of functions (f_ϵ) is bounded in the Banach space X and $m = \rho u$ is the momentum vector of the fluid. We have then used these boundedness properties, along with energy estimates, to find weak convergence of the solutions $(\rho_\epsilon, u_\epsilon, b_\epsilon)$ to a solution of a limit PDE system. The study, which heavily used Di Perna-Lions theory and compensated compactness methods after [55] and [46], featured as a special case *quasi-homogeneous densities* where $\rho_{0,\epsilon}$ was given as a perturbation of a constant density state

$$\rho_{0,\epsilon} = 1 + \epsilon r_{0,\epsilon}, \qquad \text{with } (r_{0,\epsilon})_{\epsilon>0} \subset L^\infty \cap L^2.$$

In this case, the limit system was a *quasi-homogeneous MHD system*

$$\begin{cases} \partial_t U + (U\cdot\nabla)U + RU^\perp + \nabla\left(\Pi + \frac{1}{2}|B|^2\right) = (B\cdot\nabla)B + \nu(1)\Delta U \\ \partial_t B + (U\cdot\nabla)B = (B\cdot\nabla)U + \mu(1)\Delta B \\ \partial_t R + U\cdot\nabla R = 0 \\ \operatorname{div}(U) = 0. \end{cases} \tag{56}$$

The functions (R, U, B) are the (weak) limits $(r_\epsilon, u_\epsilon, b_\epsilon) \rightharpoonup (R, U, B)$.

In Chapter 5, we pursue this study further by providing a quantitative convergence speed when strong convergence of the initial data is assumed: define the quantities $(\delta r_\varepsilon, \delta u_\varepsilon, \delta b_\varepsilon)$ by

$$\delta r_\varepsilon = r_\epsilon - R, \qquad \delta u_\varepsilon = u_\epsilon - U, \qquad \delta b_\varepsilon = b_\epsilon - B.$$

Then, by essentially subtracting the limit system (56) to the primitive equations (55), we obtain a structure theorem for the solutions $(r_\epsilon, u_\epsilon, b_\epsilon)$ at any given and finite $\epsilon > 0$. Roughly, they are equal to (R, U, B) with an additional $O(\epsilon)$ term, if μ, ν are, say, C^1. This is Theorem 5.4, which we also give below.

Theorem 5. *Let $h \equiv 1$ in* (55) *and ν, μ continuous and bounded away from zero. For a given modulus of continuity σ, assume in addition that $\nu, \mu \in C_\sigma(\mathbb{R})$. Consider a sequence $\big(\rho_{0,\varepsilon}, u_{0,\varepsilon}, b_{0,\varepsilon}\big)_{\varepsilon>0}$ of initial data satisfying the assumptions fixed in Subsection* 5.2.1*, and let $\big(\rho_\varepsilon, u_\varepsilon, b_\varepsilon\big)_{\varepsilon>0}$ be a corresponding sequence of global in time finite energy weak solutions to system* (7)*. Define $M > 0$ by*

$$M := \sup_{\varepsilon>0} \|r_{0,\varepsilon}\|_{L^\infty} + \sup_{\varepsilon>0} \|u_{0,\varepsilon}\|_{L^2} + \sup_{\varepsilon>0} \|b_{0,\varepsilon}\|_{L^2}.$$

Assume also that the triplet $\big(R_0, U_0, B_0\big)$*, defined in* (5.6)*, belongs to* $H^{1+\beta}(\Omega) \times H^1(\Omega) \times H^1(\Omega)$*, for some* $\beta \in]0,1[$ *, and let* $\big(R, U, B\big)$ *be the corresponding unique solution to system* (56)*, as given by Theorem* 5.3*. Finally, set*

$$\delta r_\varepsilon := r_\varepsilon - R\,, \qquad \delta u_\varepsilon := u_\varepsilon - U\,, \qquad \delta b_\varepsilon := b_\varepsilon - B$$

and, with analogous notation, $\delta r_{0,\varepsilon} := r_{0,\varepsilon} - R_0$*,* $\delta u_{0,\varepsilon} := u_{0,\varepsilon} - U_0$ *and* $\delta b_{0,\varepsilon} := b_{0,\varepsilon} - B_0$*.*

Then, for all fixed times $T > 0$*, the following estimate holds true: for any* $\varepsilon > 0$ *and almost every* $t \in [0,T]$*,*

$$\begin{aligned}\|\delta r_\varepsilon(t)\|_{L^2}^2 + \|\delta u_\varepsilon(t)\|_{L^2}^2 + \|\delta b_\varepsilon(t)\|_{L^2}^2 + \int_0^t \Big\{\|\nabla \delta u_\varepsilon\|_{L^2}^2 + \|\nabla \times \delta b_\varepsilon\|_{L^2}^2\Big\}\,\mathrm{d}\tau \\ \leq C\left\{\|\delta r_{0,\varepsilon}\|_{L^2}^2 + \|\delta u_{0,\varepsilon}\|_{L^2}^2 + \|\delta b_{0,\varepsilon}\|_{L^2}^2 + \max\left\{\varepsilon^2\,,\,\sigma^2(M\varepsilon)\right\}\right\},\end{aligned}$$

where the constant $C > 0$ *depends on* T*, on the lower bounds* ν_* *and* μ_* *as well as on* $|\nu|_{\mathcal{C}_\sigma}$ *and* $|\mu|_{\mathcal{C}_\sigma}$*, on the norms of the initial data* $\|u_0\|_{H^1}$*,* $\|b_0\|_{H^1}$ *and* $\|r_0\|_{H^{1+\beta}}$*, and on* M*.*

The main challenge in the proof of Theorem 5 is that simply taking the difference between (55) and the limit system (56) introduces objects whose regularity is uncertain. We therefore develop a relative entropy inequality to proceed rigorously, as was done by Feireisl, Jin and Novotný [51].

We point out that with the well-posedness results of Chapter 3, we are also able to use the relative entropy inequality so as to provide convergence in the case of vanishing viscosities $h(\epsilon) \to 0^+$. This is the object of Theorem 5.7.

Chapter 1

Littlewood-Paley Analysis and Besov Spaces

De l'absolu pouvoir vous ignorez l'ivresse,
Et des lâches flatteurs la voix enchanteresse.
Jean Racine[1]

1.1 Introduction

As we have explained in the introduction above, the study of incompressible fluids is intimately linked to that of Fourier multiplication operators. These are defined in the following way: a Fourier multiplier whose symbol is a function $\varphi(\xi)$ is the operator $\varphi(D)$ formally defined by its Fourier transform

$$\forall f \in \mathcal{S}, \qquad \widehat{\varphi(D)f}(\xi) = \varphi(\xi)\widehat{f}(\xi).$$

In particular, the case where $\varphi(\xi) = m(\xi)$ is a homogeneous function of degree zero is recurring, not to say omnipresent, in incompressible hydrodynamics.[2] Unless $m(\xi)$ is trivial, the symbol must be discontinuous at $\xi = 0$. This singularity raises many questions as to the definition of the operators $m(D)$. For instance, it is not immediately clear whether $m(D)f$ can be well-defined if the Fourier transform $\widehat{f}$ is not a measurable function, as will be the case if $f \in L^\infty$. Similarly, boundedness properties of $m(D)$ are equally non-trivial, even in L^p spaces with $p \neq 2$.

It is the purpose of this chapter to present a functional framework that is adapted to the operators $m(D)$: that of Littlewood-Paley analysis and Besov spaces. These tools have played an increasingly large role in the study of non-linear PDEs, and will constitute a dominant color in this dissertation, which warrants the dedication of a whole chapter.

Before moving onwards, we would like to make two remarks concerning the material contained below. Firstly, much of what we say about Littlewood-Paley analysis has already been written a number of times by more experienced authors (the reader will plainly see the influence of the classical textbook [7]). However, insofar as a novice's perspective allowed it, we have attempted to set some of the results of the theory into context by departing slightly from what has now become a usual presentation. Secondly, the last part of this chapter contains results concerning homogeneous Besov spaces which are our own. We hope these two features will keep the reader interested throughout this preliminary part.

The contents of this chapter are as follow: Section 1.2 is a short explanation of why frequency-annuli are vital when dealing with Fourier multipliers; Section 1.3 presents the Littlewood-Paley

[1] Jean Racine, *Athalie*, Ac. IV, Sc. III, vv. 1329-1330.

[2] An example of utmost importance is the Leray projector whose symbol is the matrix $m(\xi) = \mathrm{Id} - \frac{\xi \otimes \xi}{|\xi|^2}$.

decomposition in itself; Section 1.4 presents the definition and basic properties of Besov spaces; in Section 1.5, we introduce paradifferential calculus and Bony's decomposition; finally, in Section 1.7, we provide a lengthy discussion on Chemin's space $\mathcal{S}'_h$ of homogeneous distributions and present our results on that topic.

1.2 The Importance of Frequency-Annuli

One of the main reasons to introduce Littlewood-Paley analysis is that the usual Lebesgue and Sobolev spaces are ill-suited for the study of certain Fourier multiplication operators. This can be seen though a simple example.

Let $m(\xi)$ be a non-constant bounded homogeneous function of degree zero, we will study some of the boundedness properties of the operator $m(D)$. Now, since $m(\xi)$ is a bounded function, the Fourier-Plancherel theorem shows that $m(D)$ is bounded on L^2, as well as on all the Sobolev spaces H^s, with $s \in \mathbb{R}$. On the other hand, determining whether the same is true in L^p with $p \neq 2$ is less than obvious. In fact, it is false if $p \in \{1, +\infty\}$.

The most apparent pathology of $m(D)$ is the discontinuity of the symbol at $\xi = 0$, which has a first immediate consequence. Let $f \in L^1$, then the Fourier transform of f must be a bounded continuous function $\widehat{f} \in C_b$. However, it cannot be so for $m(D)f$, since the Fourier transform $m(\xi)\widehat{f}(\xi)$ cannot be continuous at $\xi = 0$ as soon as $\widehat{f}(0) \neq 0$.

As a consequence, the operator $m(D) : L^1 \longrightarrow \mathcal{S}'$ does not have its range in L^1. In fact, the same argument proves that $m(D)f$ cannot even be a finite measure.

Since the problem comes from the singularity of the symbol at $\xi = 0$, one may wonder whether the problem survives when the singularity is cut away. Consider a cut-off function $\chi \in \mathcal{D}$ such that $\chi \equiv 1$ around $\xi = 0$ and set

$$T = \big(\mathrm{Id} - \chi(D)\big)m(D).$$

Proposition 1.1. *Let m be as defined above. The operator T is not bounded in the $L^1 \longrightarrow \mathcal{M}$ topology, where $\mathcal{M}$ is the space of finite Borel measures.*

Proof. Assume, in order to obtain a contradiction, that $T : L^1 \longrightarrow \mathcal{M}$ is bounded. Let $f \in L^1 \cap L^2$ and define, for all $\lambda > 0$,

$$f_\lambda = \big(\mathrm{Id} - \chi(\lambda D)\big)m(\lambda D)f = \big(\mathrm{Id} - \chi(\lambda D)\big)m(D)f.$$

Firstly, if the operator T had its range in $\mathcal{M}$, then $Tf \in L^2 \cap \mathcal{M} = L^2 \cap L^1$. By scaling, we see that the family of functions $(f_\lambda)_\lambda$ is uniformly bounded in L^1.

Secondly, the functions f_λ converge to $m(D)f$ in the space L^2 as $\lambda \to +\infty$. In particular, we may fix a subsequence $f_k = f_{\lambda_k}$ which converges to f almost everywhere. Then Fatou's lemma implies that

$$\int |f| \leq \varliminf_k \int |f_k| \leq C,$$

so that $f \in L^1$. Finally, $\widehat{f}(\xi) = m(\xi)\widehat{f}(\xi)$ cannot be continuous if $\widehat{f}(0) \neq 0$, implying that $f \notin \mathcal{M}$, which is a contradiction. □

Remark 1.2. The statement of Proposition 1.1 above focuses on the Lebesgue exponent $p = 1$. The case of $p = +\infty$ is very different: to begin with, the Fourier transforms of L^∞ functions may not even be measurable functions. For example, the Fourier transforms of the constant function 1 and the sign function $\sigma = \mathbb{1}_{\mathbb{R}_+} - \mathbb{1}_{\mathbb{R}_-}$ are, respectively, the Dirac mass δ_0 and the principal value distribution p.v.$(1/\xi)$. Therefore, the product $m(\xi)\widehat{f}(\xi)$ may not even be well-defined as a distribution!

It may nevertheless be shown, using Fefferman-Stein duality, that $m(D)$ defines a bounded map $L^\infty/\mathbb{R} \longrightarrow \mathrm{BMO}$ to the space of functions of Bounded Mean Oscillations (see Section 1.6). In general, the range of $m(D)$ is not $L^\infty/\mathbb{R}$. We refer to Section 3.2 in [19] for a precise example and further discussion.

Proposition 1.1 highlights the fact that the unpleasant behavior of the operator $m(D)$ does not only come from the discontinuity of the symbol at $\xi = 0$, but also from its behavior at large frequencies. This means that to obtain a bounded operator, one must restrict the study of $m(D)$ to a definite range of frequencies that are both bounded and far from $\xi = 0$. With that in mind, we fix a function $\varphi \in C^\infty$ which is supported in an annulus

$$\mathrm{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^d, \quad \frac{1}{2} \leq |\xi| \leq 2\right\}.$$

Then, the operator $\phi(D)m(D)$ is a Fourier multiplier whose kernel lies in the Schwartz class $\mathcal{S}$, and therefore is well-defined and bounded as a $L^p \longrightarrow L^p$ operator for all $p \in [1, +\infty]$.

The basic idea behind the Littlewood-Paley decomposition, which we will define in the next Section, is to localize functions on annuli on which operators of the type $m(D)$ will be bounded in $L^p \longrightarrow L^p$ topology.

1.3 Littlewood-Paley Decompositions

The Littlewood-Paley decomposition is a partition (in fact two partitions) of the unity in the frequency space into a family of frequency-annuli whose sizes are roughly powers of 2. We fix a smooth radial function χ supported in the ball $B(0,2)$, equal to 1 in a neighborhood of $B(0,1)$ and such that $r \mapsto \chi(r\,e)$ is non-increasing over $\mathbb{R}_+$ for all unitary vectors $e \in \mathbb{R}^d$. Set $\varphi\,(\xi) = \chi\,(\xi) - \chi\,(2\xi)$ and $\varphi_m(\xi) := \varphi(2^{-m}\xi)$ for all $j \geq 0$. The functions χ and φ_j then satisfy

$$\chi(\xi) + \sum_{j\geq 0} \varphi_j(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^d \qquad \text{and} \qquad \sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1 \text{ for } \xi \neq 0. \tag{1.1}$$

In the above, the sums must be understood in the sens of pointwise convergence. Before moving on, note that the second sum has value zero at $\xi = 0$, as this fact will have important consequences.

We associate a family of Fourier multipliers to the functions χ and φ_j: the homogeneous dyadic blocks $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ are defined by Fourier multiplication, namely

$$\forall j \in \mathbb{Z}, \qquad \dot{\Delta}_j = \varphi(2^{-j}D),$$

whereas the non-homogeneous dyadic blocks $(\Delta_j)_{j\in\mathbb{Z}}$ are defined by

$$\Delta_j := 0 \quad \text{if } j \leq -2, \qquad \Delta_{-1} := \chi(D) \qquad \text{and} \qquad \Delta_j := \varphi(2^{-j}D) \quad \text{if } j \geq 0\,.$$

By virtue of (1.1), knowledge of the dyadic blocks allows us to *formally* reconstruct any function: this is the Littlewood-Paley decomposition. Formally, we have, for all f

$$f = \sum_{j\geq -1} \Delta_j f \qquad \text{and} \qquad f = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f. \tag{1.2}$$

However, although the first identity holds for $f \in \mathcal{S}'$ with convergence in that space, the second one does not. For instance, if $f = Q \in \mathbb{R}[X]$ is a nonzero polynomial function (so that the Fourier transform of Q is supported at $\xi = 0$), all the partial sums of the series are zero

$$0 = \sum_{-N}^{\infty} \dot{\Delta}_j Q \underset{N\to+\infty}{\longrightarrow} 0 \neq Q.$$

Of course, this is a plain consequence of the fact that the second identity in (1.1) is no longer true at the frequency $\xi = 0$. The question of determining whether the series $\sum_{j\in\mathbb{Z}} \dot{\Delta}_j f$ converges to f, or converges at all, will interest us in Section 1.7. For now, we simply define the space $\mathcal{S}'_h$ of distributions $f \in \mathcal{S}'$ for which the second decomposition in (1.2) holds.[3]

Definition 1.3. Recall the function χ from (1.1). Let $\mathcal{S}'_h$ be the space of those tempered distributions $u \in \mathcal{S}'$ such that

$$\chi(\lambda D)u \underset{\lambda\to+\infty}{\longrightarrow} 0 \qquad \text{in } \mathcal{S}'. \tag{1.3}$$

Note that this space does not depend on the precise choice of the low frequency cut-off χ. We refer to Section 1.7.1 lower and Lemmata 2.6 and 2.8 in chapter two for examples of $\mathcal{S}'_h$ functions.

Remark 1.4. Multiple definitions of the space $\mathcal{S}'_h$ coexist. For example, the convergence (1.3) is sometimes required to be in the norm topology of L^∞, as in [7]. We follow Section 1.5.1 in [32] and Definition 2.1.1 in [20] and give a definition which is adapted to our context. We will give other definitions and discuss their equivalence in Subsection 1.7.1.

As we have explained in Section 1.2 above, the main interest of the Littlewood-Paley decomposition is the way the dyadic blocks interact with homogeneous Fourier multipliers. Lemma 1.5 is a precise statement corresponding to that idea: a Fourier multiplier whose symbol is a homogeneous function of degree N will act on a dyadic block $\dot{\Delta}_j f$ roughly as a multiplication by 2^{Nj}, the size of the spectral support of $\dot{\Delta}_j$ being 2^j.

Lemma 1.5 (see Lemma 2.2 in [7])**.** *Let $m(\xi)$ be a C^k function away from $\xi = 0$ with $k = 2\lfloor 1 + d/2\rfloor$ and assume there is a degree $s \in \mathbb{R}$ such that $|\nabla^l m(\xi)| \le C|\xi|^{s-l}$ for all $\xi \neq 0$ and $0 \le l \le k$. Then there exists a constant depending only on m and on the the dyadic decomposition function χ such that, for all $p \in [1, +\infty]$,*

$$\forall j \in \mathbb{Z}, \forall f \in L^p, \qquad \|\dot{\Delta}_j m(D) f\|_{L^p} \le C 2^{js} \|\dot{\Delta}_j f\|_{L^p}.$$

Remark 1.6. The statement of Lemma 1.5 differs slightly from the homologous assertion in [7], Lemma 2.2 p. 53. The reason is that the constants appearing in [7] depend implicitly on the size of the annuli the functions are spectrally supported in, as our discussion in Section 1.2 makes it clear that that the result cannot hold for functions whose spectral support contain large frequencies $|\xi| \to +\infty$.

In addition to Lemma 1.5, the Bernstein inequalities show how the dyadic blocks behave under the action of derivatives. On the one hand, derivatives act pretty much as homotheties, in accordance to Lemma 1.5. On the other, the link between differentiability and integrability is made clear.

Lemma 1.7 (Bernstein inequalities, Lemma 2.1 in [7])**.** *Let $0 < r < R$. A constant C exists so that, for any nonnegative integer k, any couple (p, q) in $[1, +\infty]^2$, with $p \le q$, and any function $u \in L^p$, we have, for all $\lambda > 0$,*

$$\operatorname{Supp} \widehat{u} \subset B(0, \lambda R) \implies \|\nabla^k u\|_{L^q} \le C^{k+1} \lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_{L^p} \,;$$
$$\operatorname{Supp} \widehat{u} \subset \{\xi \in \mathbb{R}^d, \quad r\lambda \le |\xi| \le R\lambda\} \implies C^{-k-1} \lambda^k \|u\|_{L^p} \le \|\nabla^k u\|_{L^p} \le C^{k+1} \lambda^k \|u\|_{L^p}\,.$$

Remark 1.8. It appears from the proof of the first Bernstein inequality (see Lemma 2.1 in [7]) that it may be generalized to Fourier multipliers whose symbols are at least of class C^{2d}. In particular, for symbols that are homogeneous functions, we must use Lemma 1.5 above.

[3] In the notation $\mathcal{S}'_h$, the h index stands for "homogeneous", as the space $\mathcal{S}'_h$ serves as the basis for the realization of many homogeneous spaces (Sobolev, Besov, Triebel-Lizorkin, ...). This space was introduced by Jean-Yves Chemin in the mid 90s to use those realizations in the analysis of PDE problems.

Remark 1.9. Because the dyadic blocks are simply scaled versions of the Fourier multiplier $\varphi(D) = \Delta_0$, all the operators $\dot{\Delta}_j$ and Δ_j are bounded in the $L^p \longrightarrow L^p$ topology for all $p \in [1, +\infty]$, with norms independent of j and p. The same applies for the operator $S_j = \sum_{n \leq j-1} \Delta_n$, which is simply a scaled version of Δ_{-1}.

1.4 Besov Spaces

In this Section, we present the class of Besov spaces. These may be seen as *ad hoc* Banach spaces that are built on the Littlewood-Paley decomposition, and are therefore perfectly suited to Fourier multiplication operators, unlike Lebesgue spaces, which have the pathologies described in Section 1.2.

Let us introduce our discussion by giving a characterization of the usual Sobolev spaces H^s in terms of the Littlewood-Paley decomposition. For any $s \in \mathbb{R}$, the space H^s is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{H^s}^2 := \left\|(\mathrm{Id} - \Delta)^{s/2} f\right\|_{L^2}^2 = \int |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^s \mathrm{d}\xi < +\infty.$$

Now, thanks to Lemma 1.5, we see that the space H^s may be defined by an equivalent norm (noted, for the moment, $\| \, . \, \|_B$) defined in terms of the non-homogeneous Littlewood-Paley decomposition (1.2):

$$\|f\|_B := \left(\sum_{j \geq -1} \left(2^{js} \|\Delta_j f\|_{L^2}\right)^2 \right)^{1/2}.$$

The idea behind Besov spaces is simply to generalize this construction to arbitrary Lebesgue exponents $p \in [1, +\infty]$.

1.4.1 Non-Homogeneous Besov Spaces

As with Sobolev spaces, there are homogeneous and non-homogeneous Besov spaces, both having their uses and drawbacks. We start by defining the class of non-homogeneous Besov spaces, which are by far the easiest to handle while working with PDEs.

Definition 1.10. Let $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$. The *non-homogeneous Besov space* $B^s_{p,r} = B^s_{p,r}(\mathbb{R}^d)$ is defined as the set of tempered distributions $u \in \mathcal{S}'$ for which

$$\|f\|_{B^s_{p,r}} := \left\| \left(2^{js} \, \|\Delta_j f\|_{L^p}\right)_{j \geq -1} \right\|_{\ell^r} < +\infty \, .$$

The non-homogeneous space $B^s_{p,r}$ is Banach for all values of (s, p, r).

As with the usual (potential) Sobolev spaces $W^{s,p}$, the exponent $s \in \mathbb{R}$ in the Besov space $B^s_{p,r}$ acts as a regularity index. In fact, it is a consequence of Lemma 1.7 that there is equivalence of norms

$$\forall f \in B^s_{p,r}, \qquad \|f\|_{B^s_{p,r}} \approx \left\|(\mathrm{Id} - \Delta)^{s/2} f\right\|_{B^0_{p,r}}.$$

In particular, if $s < \sigma$, then $B^\sigma_{p,r} \subsetneq B^s_{p,r}$. In fact, the Besov space $B^s_{p,r}$ has a close behavior to that of the Sobolev space $W^{s,p}$. We will examine this comparison in more detail below.[4]

As an illustration of how Besov spaces may find their usefulness, we will show a simple bound for the Fourier multiplier $m(D)$, where, as usual, $m(\xi)$ is a bounded homogeneous function of degree zero. More precisely, we show that for all $s \in \mathbb{R}$, $r \in [1, +\infty]$ and $p \in [2, +\infty]$, we have

$$\forall f \in L^2 \cap B^s_{p,r}, \qquad \|m(D) f\|_{B^s_{p,r}} \leq C \|f\|_{L^2 \cap B^s_{p,r}}. \tag{1.4}$$

[4] Besov spaces can also be seen as interpolation spaces between (potential) Sobolev spaces, see [77] Definition 3.4.

Proof of (1.4). Firstly, we decompose the $B^s_{p,r}$ norm into low frequencies, associated to the block Δ_{-1}, and high frequencies: in the inequality below, the sum is to be replaced by the usual ℓ^∞ norm if $r = +\infty$,

$$\|m(D)f\|_{B^s_{p,r}} \leq \|\Delta_{-1} m(D)f\|_{L^p} + \left(\sum_{j \geq 0} 2^{jsr} \|\Delta_j m(D)f\|^r_{L^p} \right)^{1/r}.$$

We start by focusing on the low frequencies. Since $p \geq 2$, the first Bernstein inequality (Lemma 1.7) and the fact that $m(\xi)$ is bounded provide

$$\|\Delta_{-1} m(D)f\|_{L^p} \leq C \|\Delta_{-1} m(D)f\|_{L^2} = \left(\int \left| \chi(\xi) m(\xi) \widehat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} \leq \|f\|_{L^2}.$$

As for the high frequencies, Lemma 1.5 shows that all the norms $\|\Delta_j m(D)f\|_{L^p}$ are bounded by $\|\Delta_j f\|_{L^p}$. This with the low frequency inequality immediately above gives the sought inequality. □

The careful reader will have noticed that the L^2 assumption in the proof above is useful only for dealing with the low frequency part of $m(D)f$. This is the most glaring shortcoming of *non-homogeneous* Besov spaces: although the Littlewood-Paley decomposition is well-suited for dealing with frequencies that are far away from $\xi = 0$, which will lie in various dyadic annuli, we are left to our own devices for working with the Δ_{-1} part. This will either require a strong integrability assumption to make use of the first Bernstein inequality ($f \in L^2$ in the case of (1.4), although $f \in L^q$ with $q < p$ would suffice) or resorting to more sophisticated analysis, such as Calderón-Zygmund theory.

Although this will mainly be accessory, let us mention for the sake of completeness that $m(D)$ can be seen as a singular integral operator, so it is bounded in the $L^p \longrightarrow L^p$ topology for $1 < p < +\infty$ (see Proposition 2, Section 4.4 pp. 245-247 in [100]). As we have explained in Section 1.2, this result collapses for $p \in \{1, +\infty\}$, although Fefferman-Stein duality makes it possible to study the operator in the space BMO of functions of Bounded Mean Oscillations (see Section 1.6 below).

As the index p and the exponent s of the Besov space $B^s_{p,r}$ play a similar role those in the Sobolev space $W^{s,p}$, a comparison between both class of spaces is very natural, not to mention its constant usefulness in the rest of this dissertation. Though much can be said, we will limit ourselves to a few recurring (and elementary) examples.

First of all, as we have explained in the beginning of this Section, the spaces H^s and $B^s_{2,2}$ are equal for all $s \in \mathbb{R}$, their norms being quasi-isometrical.

Next, for all $p \in [1, +\infty]$, we consider the case of L^p and where it stands with respect to Besov spaces. First of all, if $f \in B^0_{p,1}$, then the series $f = \sum_{j \geq -1} \Delta_j f$ actually converges in the norm topology of L^p, so that $f \in L^p$. Secondly, if $f \in L^p$, then the functions $\Delta_j f$ are uniformly bounded in L^p with respect to j, thus $f \in B^0_{p,\infty}$. We deduce the chain of embeddings

$$B^0_{p,1} \xrightarrow{\subset} L^p \xrightarrow{\subset} B^0_{p,\infty}.$$

A way to read the first Bernstein inequality is to compare it with the usual Sobolev embeddings, which have the same scaling. In fact, this inequality translates into embeddings of Besov spaces: if $p \leq q$ and $r \leq t$, then for all $s \in \mathbb{R}$ we have $B^s_{p,r} \subset B^{s-d(1/p-1/q)}_{q,t}$. In particular, the spaces $B^{d/p}_{p,1}$ are contained in L^∞ through the chain of embeddings

$$B^{d/p}_{p,1} \xrightarrow{\subset} B^0_{\infty,1} \xrightarrow{\subset} C_b \xrightarrow{\subset} L^\infty \xrightarrow{\subset} B^0_{\infty,\infty}. \tag{1.5}$$

In the above, C_b is the set of continuous and bounded functions. The fact that $B^0_{\infty,1} \subset C_b$ simply stems from the fact that the series $\sum_j \Delta_j f$ is normally convergent if $f \in B^0_{\infty,1}$. Similarly, we have

$$B^{1+d/p}_{p,1} \xrightarrow{\subset} B^1_{\infty,1} \xrightarrow{\subset} C^1_b \xrightarrow{\subset} W^{1,\infty} \xrightarrow{\subset} B^1_{\infty,\infty}.$$

For this reason, the spaces $B^{d/p}_{p,1}$ (respectively $B^{1+d/p}_{p,1}$) are sometimes called *critical*, in the sense that they are somewhat optimal in the class of Besov spaces with respect to the embedding in L^∞ (resp. $W^{1,\infty}$). Spaces $B^s_{p,r}$ such that $B^s_{p,r} \subset W^{1,\infty}$ will be called *Besov-Lipschitz spaces*, or *supercritical* Besov spaces, in the sense that their regularity is above the critical value.

We end this Section on non-homogeneous Besov spaces by a few concluding remarks.

Remark 1.11. First of all, we note that all the embeddings appearing in (1.5) are *strict* if $p < +\infty$. Firstly, the constant function 1 is in all the spaces except $B^{d/p}_{p,1}$. Next, the space $B^1_{\infty,1}$ is embedded in the space C_u of bounded uniformly continuous functions. Let us prove that assertion. Consider a $f \in B^1_{\infty,1}$ and a $\epsilon > 0$, and let $N \geq -1$ to be fixed later on. For all $x, y \in \mathbb{R}^d$, we have

$$|f(x) - f(y)| \leq \sum_{j=-1}^{N} |\Delta_j f(x) - \Delta_j f(y)| + 2 \sum_{j=N+1}^{\infty} \|\Delta_j f\|_{L^\infty}$$

Because $f \in B^1_{\infty,1}$ we may chose N large enough that the second sum is bounded by ϵ. On the other hand, we bound the terms of the first sum by the mean value theorem and the first Bernstein inequality (Lemma 1.7), giving

$$|f(x) - f(y)| \leq \epsilon + |x - y| \sum_{j=-1}^{N} \|\nabla \Delta_j f\|_{L^\infty} \leq \epsilon + |x - y| \sum_{j=-1}^{N} 2^j \|\Delta_j f\|_{L^\infty}.$$

Since N is now fixed, we may chose a $\delta > 0$ such that the whole righthand side is smaller than 2ϵ if $|x - y| \leq \delta$, thus showing that f is indeed uniformly continuous. Finally, the function

$$f(x) = \sum_{j \geq -1} \exp(2^j x),$$

where the sum converges in the $\mathcal{S}'$ topology, is an element of $B^0_{\infty,\infty}$ which is clearly not in L^∞. If it were, f would define a function on the torus $f \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$ whose Fourier series would converge in mean quadratic norm.

Remark 1.12. The embedding properties of Besov spaces are one of their major advantages in comparaison to Sobolev spaces. For instance, it is not true that critical Sobolev spaces $H^{d/2}$ are included in L^∞ (in fact, we have $H^{d/2} \hookrightarrow \mathrm{BMO}$, see Theorem 1.48 in [7] and the embeddings of Section 1.6), whereas the (slightly smaller) critical space $B^{d/2}_{2,1}$ is in L^∞. This nice property of the Besov spaces come at a cost: the critical Sobolev spaces $W^{d/p,p}$ are reflexive if $1 < p < +\infty$, but the critical Besov spaces $B^{d/p}_{p,1}$ are not.

Remark 1.13. It may seem at first glance that the regularity exponent s plays a much more important role in the structure of $B^s_{p,r}$ as the index r. For example, if $\sigma < s$ then we have $B^s_{p,\infty} \subset B^\sigma_{p,1}$. However, this is slightly misleading: for any given $r, p \in [1, +\infty]$ and $s, \sigma \in \mathbb{R}$, the spaces $B^s_{p,r}$ and $B^\sigma_{p,r}$ are (quasi) isometrical, whereas if $p < +\infty$ then $B^s_{p,r}$ is separable if and only if $r < +\infty$. We do not comment further on this, as we will give additional details on how different these spaces are in Section 1.7.

1.4.2 Homogeneous Besov Spaces

In this paragraph, we focus on homogeneous Besov spaces, which are built after the homogeneous Littlewood-Paley decomposition. This part, which is more technical than the previous one, is mainly here to prepare for the ideas introduced in Section 1.7.

As we have noted when dealing with non-homogeneous Besov spaces, their main issue is that the low frequency block Δ_{-1} of a function is not spectrally supported in an annulus. This creates difficulties when dealing with Fourier multipliers such as in (1.4), which are precisely singular at $\xi = 0$. The idea of *homogeneous* Besov spaces $\dot{B}^s_{p,r}$ is to simply use the homogeneous Littlewood-Paley decomposition to define a norm: for all $s \in \mathbb{R}$ and $p, r \in [1, +\infty]$ set

$$\|f\|_{\dot{B}^s_{p,r}} := \left\| \left(2^{ms} \|\dot{\Delta}_m f\|_{L^p} \right)_{m \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}. \tag{1.6}$$

Now, thanks to Lemma 1.5, it is obvious that the operator $m(D)$, where $m(\xi)$ is again bounded and homogeneous of degree zero, will fulfill $\|m(D)f\|_{\dot{B}^s_{p,r}} \leq C\|f\|_{\dot{B}^s_{p,r}}$. But we cannot be satisfied yet: (1.6) does not define in itself a Banach space because it is not a norm, as it vanishes on the set $\mathbb{R}[X]$ of polynomial functions. As we have already pointed out above, this degeneracy of the semi-norm $\|\,.\,\|_{\dot{B}^s_{p,r}}$ is linked to the fact that the homogeneous Littlewood-Paley decomposition fails for polynomials. In order to circumvent this problem, the appropriate answer is to simply work modulo polynomials.

Definition 1.14. Let $1 \leq p, r \leq +\infty$ and $s \in \mathbb{R}$. We define the *homogeneous* Besov space $\dot{B}^s_{p,r}$ as being the set of $f \in \mathcal{S}'/\,\mathbb{R}[X]$ such that $\|f\|_{\dot{B}^s_{p,r}} < +\infty$. These spaces are complete (Theorem 3.19 in [93]).

As we have said, the homogeneous Besov spaces behave better under the action of homogeneous Fourier multipliers than their non-homogeneous counterpart. For example, the fractional Laplacian defines an isomorphism between all spaces with same indices p and r (see also Theorem 3.17 in [93]).

Proposition 1.15. *Let $s, \sigma \in \mathbb{R}$ and $p, r \in [1, +\infty]$. Then the fractional Laplace operator defines an isomorphism of Banach spaces $(-\Delta)^{\sigma/2} : \dot{B}^s_{p,r} \longrightarrow \dot{B}^{s-\sigma}_{p,r}$.*

Proof. The isometry property is an immediate consequence of Lemma 1.5, the homogeneous norm is built for that purpose. The non-trivial part of the proof is showing that if $f \in \dot{B}^s_{p,r}$, then $(-\Delta)^{\sigma/2} f$ indeed defines a tempered distribution modulo a polynomial. We will work in Fourier variables: we seek a tempered distribution $T \in \mathcal{S}'$ such that $\widehat{T}(\xi) = |\xi|^\sigma \widehat{f}(\xi)$ in $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$, so that T will define a unique element in $\mathcal{S}'/\,\mathbb{R}[X]$.

First of all, we must see that there is no problem for frequencies that are away from $\xi = 0$ because $|\xi|^\sigma$ is smooth on C^∞. To separate low and high frequencies, we fix a $\chi \in \mathcal{D}$ such that $\chi(\xi) \equiv 1$ in a neighborhood of the origin. Let $\phi \in \mathcal{S}$ be a Schwartz function, and decompose its Fourier transform into

$$\widehat{\phi}(\xi) \;=\; \chi(\xi)\,\widehat{\phi}(\xi) \;+\; \big(1 - \chi(\xi)\big)\,\widehat{\phi}(\xi)\,.$$

Observe that $|\xi|^\sigma(1-\chi(\xi))\phi(\xi)$ is also a Schwartz function. As for the low frequency part $\chi(\xi)\,\widehat{\phi}(\xi)$, the idea is to "flatten" the test function around $\xi = 0$ by subtracting a Taylor polynomial: let us introduce, for some integer $N \geq 0$ which we will fix later on, the ξ-function

$$\mathfrak{R}[\phi](\xi) \;:=\; \chi(\xi)\, \left(\widehat{\phi}(\xi) \;-\; \sum_{|\alpha| \leq N} \frac{\xi^\alpha}{\alpha!} \partial^\alpha \widehat{\phi}(0) \right).$$

The sum ranges on all multi-indices $\alpha \in \mathbb{N}^d$ such that $|\alpha| = \alpha_1 + \ldots + \alpha_d \leq N$. Then, for N large enough, $\Phi[\phi](\xi) := |\xi|^\sigma \mathfrak{R}[\phi](\xi)$ is of class C^p, where p is the order of $\widehat{f}$ as a distribution on the compact set $\operatorname{supp}(\chi)$, so that the bracket $\langle \widehat{f}, \Phi\rangle_{\mathcal{S}'\times\mathcal{S}}$ is well-defined. Let us focus for a while on the action of $\widehat{f}$ on the polynomial part of $\widehat{\phi}$: we have

(1.7)
$$\begin{aligned}\left\langle \widehat{f}(\xi)\,,\, \chi(\xi) \sum_{|\alpha|\le N} \frac{\xi^\alpha}{\alpha!}\partial^\alpha\widehat{\phi}(0)\right\rangle_{\mathcal{S}'\times\mathcal{S}} &= \left\langle \sum_{|\alpha|\le N} \frac{1}{\alpha!}\left\langle \widehat{f}(\xi)\,,\, \xi^\alpha\,\chi(\xi)\right\rangle_{\mathcal{S}'\times\mathcal{S}} \partial^\alpha\delta_0(\xi)\,,\, \widehat{\phi}(\xi)\right\rangle_{\mathcal{S}'\times\mathcal{S}} \\ &:= \left\langle \sum_{|\alpha|\le N} \gamma_\alpha\,\partial^\alpha\delta_0(\xi)\,,\, \widehat{\phi}(\xi)\right\rangle_{\mathcal{S}'\times\mathcal{S}},\end{aligned}$$

where δ_0 is the Dirac mass centered at the origin. As a linear combination of δ_0 and its derivatives is the Fourier transform of a polynomial, we can fix $Q\in\mathbb{R}[X]$ such that the brackets (1.7) are equal to $\langle Q\,,\,\phi\rangle_{\mathcal{S}'\times\mathcal{S}}$. Hence, we have shown that

$$\langle f\,,\,\chi(D)\,\phi\rangle \;=\; \left\langle \widehat{f}\,,\,\mathfrak{R}[\phi]\right\rangle_{\mathcal{S}'\times\mathcal{S}} \;+\; \langle Q\,,\,\phi\rangle_{\mathcal{S}'\times\mathcal{S}}\,.$$

The bracket $\langle \widehat{f}\,,\,\Phi[\phi]\rangle_{\mathcal{S}'\times\mathcal{S}} = \langle |\xi|^\sigma\widehat{f}\,,\,\mathfrak{R}[\phi]\rangle_{\mathcal{S}'\times\mathcal{S}}$ is a continuous linear quantity with respect to $\phi\in\mathcal{S}$, and hence defines a tempered distribution $T\in\mathcal{S}'$ that fulfills the property

$$\widehat{T}(\xi) = |\xi|^\sigma\widehat{f}(\xi) \qquad \text{in } \mathcal{D}'(\mathbb{R}^d\setminus\{0\}),$$

which is exactly what we needed to prove. $\square$

We proceed to giving basic embedding properties of the homogeneous Besov spaces. As we will see, these ressemble closely those obtained in the non-homogeneous case, even though the homogeneous setting introduces a few differences.

First of all, as in the non-homogeneous case, the spaces $\dot{B}^s_{2,2}$ are quasi-isometrical to the homogeneous Sobolev spaces $\dot{H}^s$ of distributions $f\in\mathcal{S}'/\mathbb{R}[X]$ such that $\widehat{f}\in L^1_{\mathrm{loc}}(\mathbb{R}^d\setminus\{0\})$ and[5]

$$\left\|(-\Delta)^{s/2}f\right\|^2_{L^2} = \int |\widehat{f}(\xi)|^2|\xi|^{2s}\mathrm{d}\xi < +\infty.$$

Next, the second Bernstein inequality implies that for any $s\in\mathbb{R}$, $r\le t$ and $p\le q$ we have inclusion between the Besov spaces $\dot{B}^s_{p,r}\subset\dot{B}^{s-d(1/p-1/q)}_{q,t}$. However, it must be noted that, unlike for non-homogeneous spaces, larger regularity indices do not mean smaller spaces. If $s<\sigma$ and $p,r\in[1,+\infty]$, then the spaces $\dot{B}^s_{p,r}$ and $\dot{B}^\sigma_{p,r}$ are not comparable. This is because the homogeneous norms carry strong information with regards to the *low frequency* behavior of functions. The same phenomenon occurs for the spaces $\dot{H}^s$ and $\dot{H}^\sigma$, which are not comparable.

Homogeneous Besov spaces also compare to Lebesgue spaces: for any $p\in[1,+\infty]$ and $f\in\dot{B}^0_{p,1}$, the series $\sum_{j\in\mathbb{Z}}\dot{\Delta}_jf$ is convergent in the norm topology of L^p, thus every $f\in\dot{B}^0_{p,1}$ defines an element of L^p. And if $f\in L^p$, the functions $\dot{\Delta}_jf$ are uniformly bounded in L^p so that f defines an element of $\dot{B}^0_{p,\infty}$. Caution must used at this point: the first arrow in the chain

$$\dot{B}^0_{p,1}\longrightarrow L^p\longrightarrow\dot{B}^0_{p,\infty}.$$

is a true embedding, whereas the second may not be one. If $p<+\infty$, then the space L^p contains no non-trivial polynomial, and hence embeds in $L^p\hookrightarrow\mathcal{S}'/\mathbb{R}[X]$. However, if $p=+\infty$, then all

[5] If $\Omega\subset\mathbb{R}^d$, the space $L^1_{\mathrm{loc}}(\Omega)$ is the space of measurable functions on Ω that are integrable on every compact subset of Ω. Thus $L^1_{\mathrm{loc}}(\mathbb{R}^d\setminus\{0\})$ is different from $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

constant functions are in L^∞, so that the second arrow does in fact have a dimension one kernel.[6] In that case, one should rather write

$$\dot{B}^0_{\infty,1} \longrightarrow L^\infty \twoheadrightarrow L^\infty/\,\mathbb{R} \xrightarrow{\subset} \dot{B}^0_{\infty,\infty}. \tag{1.8}$$

In the previous example, we see that some homogeneous Besov spaces embed in $\mathcal{S}'$. This process is called *realizing* the Besov space as a subspace of $\mathcal{S}'$. A realization of the space $\dot{B}^s_{p,r}$ is a linear map $\sigma : \dot{B}^s_{p,r} \longrightarrow E$ to a subspace $E \subset \mathcal{S}'$ such that the following diagram commutes:

$$\begin{tikzcd} \dot{B}^s_{p,r} \arrow[r, "\subset"] \arrow[d, "\sigma"'] & \mathcal{S}'/\,\mathbb{R}[X] \\ E \arrow[ur, "\pi"'] & \end{tikzcd}$$

where in the above $\pi : E \subset \mathcal{S}' \longrightarrow \mathcal{S}'/\,\mathbb{R}[X]$ is the natural projection. We refer to [13] for a thorough discussion of this topic. We will simply point out that some Besov spaces can be easily realized as subspaces of $\mathcal{S}'$. Let $s \in \mathbb{R}$ and $p, r \in [1, +\infty]$ such that

$$s < \frac{d}{p} \qquad \text{or} \qquad s = \frac{d}{p} \text{ and } r = 1. \tag{1.9}$$

Then the space $\dot{B}^s_{p,r}$ is included in $\dot{B}^{s-d/p}_{\infty,1}$, so that the series $\sum_{j\leq 0} \dot{\Delta}_j f$ converges in L^∞ for all $f \in \dot{B}^s_{p,r}$. In particular, the Littlewood-Paley decomposition $g = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f$ converges in the $\mathcal{S}'$ topology to a function $g \in \mathcal{S}'_h$ (see Definition 1.3) which is equal to f up to a polynomial. This natural realization put the spaces $\dot{B}^s_{p,r}$ with indices satisfying (1.9) at a privileged place in the study of PDEs: since solving PDE problems in spaces of distributions modulo polynomials is rather uncomfortable, it is common to work instead in the space

$$\dot{\mathfrak{B}}^s_{p,r} = \Big\{ f \in \mathcal{S}'_h, \quad \|f\|_{\dot{B}^s_{p,r}} < +\infty \Big\}. \tag{1.10}$$

However, this argument allowing us to see $\dot{\mathfrak{B}}^s_{p,r}$ as a realization of $\dot{B}^s_{p,r}$ only works under condition (1.9). In fact, as we will see later, the space $\dot{\mathfrak{B}}^s_{p,r}$ ceases to be complete if (1.9) no longer holds, so it cannot define a realization of $\dot{B}^s_{p,r}$. We explore this issue in great detail in Section 1.7. The homogeneous Besov spaces $\dot{B}^s_{p,r}$ with (1.9) are said to be *subcritical*, the space $\dot{B}^{d/p}_{p,1}$ being *critical*. When (1.9) is not satisfied, $\dot{B}^s_{p,r}$ is *supercritical*.

Remark 1.16. As in the non-homogeneous case, the first and third arrows in (1.8) are strict embeddings. The same arguments apply here. In addition, we make the following remark, which is interesting in itself: the Fourier transform defines a embedding (see Theorem 3.21 in [93] for a similar result)

$$\mathcal{F} : \mathcal{M} \longrightarrow \dot{B}^0_{\infty,1} \oplus \mathbb{C}.$$

Indeed, if $\mu \in \mathcal{M}$ is a finite measure, then,

$$\sum_{-\infty}^{\infty} \|\dot{\Delta}_j \mu\|_{L^\infty} \leq \sum_{-\infty}^{\infty} \int |\varphi_j(\xi)| \, \mathrm{d}|\mu|(\xi) \leq C \sum_{-\infty}^{\infty} |\mu|\big(\mathrm{supp}(\varphi_j)\big),$$

so that $\mathcal{F} : \mathcal{M} \longrightarrow \dot{B}^0_{\infty,1}$ is a bounded map whose kernel must be made of measures supported at $\xi = 0$.

[6] Here, we are more precise than Proposition 2.39 of [7], which states that L^∞ embeds in $\dot{B}^0_{\infty,\infty}$. Although homogeneous Besov spaces are defined differently in [7], constant functions still are an issue.

To end this Section on homogeneous Besov spaces, we give the duality properties which the Besov spaces inherit from the Lebesgue ones. We refer to [89], Theorem 12 in Chapter 3 pp. 74-75 for a proof.[7]

Theorem 1.17. *Let $s \in \mathbb{R}$ and $p, r \in [1, +\infty[$. Then the topological dual of $\dot{B}^s_{p,r}$ is (quasi) isometric to $\dot{B}^{-s}_{p',r'}$, where p' and r' are the conjugated exponents of p and r.*

Remark 1.18. Theorem 1.17 is analogous to Proposition 2.29 in [7]. However, [7] only works with the spaces $\mathfrak{B}^s_{p,r}$ from (1.10), even when the indices (s, p, r) do not fulfill the subcriticality condition (1.9).

Remark 1.19. In particular, if $p > 1$, then $\dot{B}^s_{p,r}$ has a separable predual. We will use this fact below.

1.5 Paradifferential Calculus

The goal of this Section is to introduce the tools we will need to study *non-linear* problems with Besov spaces. The problem is the following: given two functions f and g which have some (Besov) regularity, can the pointwise product fg be defined, and if so, what is its (Besov) regularity? The main tool to answer this question is the Bony paraproduct decomposition, after [12].[8]

Bony's decomposition is, in some very loose way, comparable to Leibniz product formula for derivatives: for example, the second derivative of a product fg reads

$$(fg)'' = gf'' + 2f'g' + fg''.$$

We make the following observation: in the total derivative $(fg)''$, the high order derivatives f'' and g'' only multiply low order terms f and g, while the intermediate order ones are mixed in the product $2f'g'$. This gross intuition of seeing how derivatives mix in a product is at the basis of Bony's decomposition.

Let $f, g \in \mathcal{S}$. By *formally* writing the non-homogeneous Littlewood-Paley decomposition for f and g, we get

$$fg = \sum_{m,n \geq -1} \Delta_m f . \Delta_n g. \tag{1.11}$$

The Fourier transform of product $\Delta_m f . \Delta_n g$ of the dyadic blocks is the convolution $(\varphi_m \widehat{f}) * (\varphi_n \widehat{g})$, and is therefore is supported in the sum of the supports of φ_m and φ_n, which are annuli of respective size 2^m and 2^n. One of three things may happen:

1. either $m \ll n$ and $(\varphi_m \widehat{f}) * (\varphi_n \widehat{g})$ is supported in an annulus of size roughly 2^n, so the regularity of g is much more important than that of f;

2. either $n \ll m$ and the reverse is true;

3. or $n \simeq m$ and the product $(\varphi_m \widehat{f}) * (\varphi_n \widehat{g})$ is supported in a ball of size roughly 2^m. Both f and g will be instrumental in determining the regularity of this part.

[7]The proof is stated in the framework of non-homogeneous spaces. We believe the arguments are the same.

[8]We only present the non-homogeneous paradifferential calculus, the homogeneous version being unused in the rest of the dissertation.

With that in mind, we decompose the product (1.11) into three parts: this is Bony's paraproduct decomposition

$$\begin{aligned} fg &= \Big(\sum_{n\le m-2} + \sum_{m\le n-2} + \sum_{|m-n|\le 1} \Big) \Delta_m f . \Delta_n g \\ &= \sum_{m\ge -1} S_{m-1} g . \Delta_m f + \sum_{n\ge -1} S_{n-1} f . \Delta_n g + \sum_{|m-n|\le 1} \Delta_m f . \Delta_n g \\ &:= \mathcal{T}_g(f) + \mathcal{T}_f(g) + \mathcal{R}(f,g), \end{aligned}$$

where the sum operator S_{m-1} is defined as $S_{m-1} = \sum_{n\le m-2} \Delta_n$ (see Remark 1.9). The operators $\mathcal{T}_f$ and $\mathcal{T}_g$ are called *paraproducts* while $\mathcal{R}$ is the *remainder*. The paraproducts are sums of terms which are spectrally supported in annuli and the remainder is a sum of functions which are spectrally supported in increasingly large balls.

When estimating the regularity of a paraproduct, say $\mathcal{T}_g(f)$, in a Besov space, one studies the size of the block $\Delta_j \mathcal{T}_g(f)$. Here, the property $n \ll m$ translates as the cancellation of most of the terms in the paraproduct when the block Δ_j is applied:

$$\Delta_j \mathcal{T}_g(f) = \Delta_j \sum_{|m-j|\le 4} S_{m-1} g . \Delta_m f.$$

This means that, very often, the regularity of $\mathcal{T}_g(f)$ will depend mainly[9] on f. For example, assume that $g \in L^\infty$ and that f is in some Besov space $B^s_{p,r}$. Then

$$\begin{aligned} \|\Delta_j \mathcal{T}_g(f)\|_{L^p} &\lesssim \sum_{|j-m|\le 4} \|S_{m-1} g\|_{L^\infty} \|\Delta_m f\|_{L^p} \\ &\lesssim \|g\|_{L^\infty} 2^{-js} \|f\|_{B^s_{p,r}} c_j, \end{aligned}$$

where $(c_j)_{j\ge -1}$ is a sequence in the unit ball of $\ell^r(j \ge -1)$. As a consequence, the paraproduct $\mathcal{T}_g(f)$ also is an element of $B^s_{p,r}$. Proposition 1.20 is a precise form of this idea.

Proposition 1.20 (see Theorem 2.82 in [7])**.** *For any $(s,p,r) \in \mathbb{R} \times [1,+\infty]^2$ and $t > 0$, the paraproduct operator $\mathcal{T}$ maps continuously $L^\infty \times B^s_{p,r}$ in $B^s_{p,r}$ and $B^{-t}_{\infty,\infty} \times B^s_{p,r}$ in $B^{s-t}_{p,r}$. Moreover, the following estimates hold:*

$$\|\mathcal{T}_u(v)\|_{B^s_{p,r}} \le C \, \|u\|_{L^\infty} \, \|\nabla v\|_{B^{s-1}_{p,r}} \qquad \text{and} \qquad \|\mathcal{T}_u(v)\|_{B^{s-t}_{p,r}} \le C \|u\|_{B^{-t}_{\infty,\infty}} \, \|\nabla v\|_{B^{s-1}_{p,r}} \, .$$

The remainder term is more difficult to handle. In fact, while the paraproduct involving two functions in arbitrary non-homogeneous Besov spaces can always be defined, it is not so for the remainder. This is of course related to the product being ill defined in general spaces of distributions. Continuity properties of the remainder are given in Proposition 1.21.

Proposition 1.21 (Theorem 2.85 in [7])**.** *For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R} \times [1,+\infty]^2$ such that $s_1 + s_2 > 0$, $1/p := 1/p_1 + 1/p_2 \le 1$ and $1/r := 1/r_1 + 1/r_2 \le 1$, the remainder operator $\mathcal{R}$ maps continuously $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ into $B^{s_1+s_2}_{p,r}$. In the case $s_1 + s_2 = 0$, provided $r = 1$, the operator $\mathcal{R}$ is continuous from $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ with values in $B^0_{p,\infty}$.*

Remark 1.22. The regularity condition $s_1 + s_2 \ge 0$ in Proposition 1.21 is of vital importance: when it is not fulfilled, the product of two functions can be absolutely wild. For instance, if

[9] As is apparent in Proposition 1.20 below, if g has negative regularity, then the general regularity of the paraproduct $\mathcal{T}_g(f)$ will suffer.

$s+t<0$, it is possible to find sequences of functions (f_n) and (g_n) which are respectively bounded in H^s and H^t but such that $\|f_n\, g_n\|_{H^{-m}} \longrightarrow +\infty$ for any $m \in \mathbb{R}$: define f_n and g_n by

$$\widehat{f_n}(\xi) = \sum_{j=1}^{n} \frac{1}{j} 2^{-jd/2} 2^{-js} \varphi_j(\xi) \qquad \text{and} \qquad \widehat{g_n}(\xi) = \sum_{j=1}^{n} \frac{1}{j} 2^{-jd/2} 2^{-jt} \varphi_j(\xi),$$

where φ_j is the Littlewood-Paley decomposition function defined in Section 1.3. Then, by observing that $\varphi_j * \varphi_j(\xi) = 2^{jd} \varphi * \varphi\,(2^{-j}\xi)$, and taking advantage of the fact that all the terms in the Fourier transforms of both f_n and g_n are nonnegative, we see that

$$\widehat{f_n} * \widehat{g_n}(\xi) \geq \sum_{j=1}^{n} \frac{1}{j^2} 2^{-j(s+t)} \varphi * \varphi\,(2^{-j}\xi).$$

Because φ is a nonzero radial function, the support of φ is a ball. Therefore, we may fix a $\delta > 0$ such that $\varphi * \varphi\,(2^{-j}\xi) \geq \delta$ for all $|\xi| \leq 2^j$, so that the sum above can be made as large as desired, since $s+t<0$. Finally, the Fourier-Plancherel theorem yields

$$\|f_n\, g_n\|_{H^{-m}} \geq \|\Delta_{-1}(f_n\, g_n)\|_{L^2} \geq C\delta 2^{-n(s+t)} \underset{n\to+\infty}{\longrightarrow} +\infty,$$

and this ends the proof.

A straightforward but useful consequence of Propositions 1.20 and 1.21 is that the subcritical[10] Besov spaces $B^s_{p,r}$ are Banach algebras for $s>0$, as can be seen by the so-called *tame estimates*: let $s \in \mathbb{R}$ and $p, r \in [1, +\infty]$ be such that $s > d/p$ or $s = d/p > 0$ and $r = 1$, then (see Corollary 2.86 in [7])

$$\|fg\|_{B^s_{p,r}} \leq \frac{C^{s+1}}{s}\Big(\|f\|_{B^s_{p,r}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{B^s_{p,r}} \Big) \leq C(s) \|f\|_{B^s_{p,r}} \|g\|_{B^s_{p,r}}. \tag{1.12}$$

Note that the endpoint space $B^0_{\infty,1}$ is not an algebra.

1.6 Overview of Singular Integral Operators

In this final preparatory Section, we give a quick summary of more advanced results concerning Singular Integrals and the space BMO of functions of Bounded Mean Oscillations. In addition to providing extra context, we resort to some of these results in the next Chapters. Our goal is not to give a complete overview of the topic (we wish to focus on Besov spaces), so the discussion will necessarily be brief.

Consider a Fourier multiplier $T = m(D)$ whose symbol is, again, a bounded homogeneous function of degree zero. As we have seen above, T defines a bounded operator $T : L^p \longrightarrow \dot{B}^0_{p,\infty}$. However, this result is not quite optimal: it is possible to be slightly more precise concerning the range of T by introducing more advanced material. We start by the classical Calderón-Zygmund estimate for Singular Integral Operators.

Theorem 1.23 (see Proposition 4 p. 250 in [100])**.** *Consider $1 < p < +\infty$. Then $T : \mathcal{S} \longrightarrow \mathcal{S}'$ has a unique bounded extension $T : L^p \longrightarrow L^p$.*

Remark 1.24. In particular, if $1 < p < +\infty$ and (s, r) are such that $s \in \mathbb{R}$ and $r \in 1, +\infty]$, then the T is continuous on *non-homogeneous* Besov spaces: it defines a bounded operator in the $B^s_{p,r} \longrightarrow B^s_{p,r}$ topology.

[10]Subcritical spaces are those that have lower regularity than critical spaces.

As explained above, this boundedness result fails in the endpoint cases $p = 1$ or $p = +\infty$. But even in those cases, it is still possible to improve the range of T by defining new spaces which will substitute for L^1 and L^∞.

Definition 1.25 (Hardy spaces and BMO)**.** Define the Hardy space $\mathcal{H}^1$ as the set of all $f \in \mathcal{S}'_h$ such that

$$\|f\|_{\mathcal{H}^1} := \left\| \Big(\sum_{j\in\mathbb{Z}} \big|\dot\Delta_j f\big|^2 \Big)^{1/2} \right\|_{L^1} < +\infty,$$

and define the space BMO of functions of Bounded Mean Oscillations as being the set of locally integrable functions modulo constants $f \in L^1_{\mathrm{loc}}/\mathbb{R}$ such that

$$\|f\|_{\mathrm{BMO}} := \sup_B \frac{1}{|B|} \int_B |f(x) - (f)_B| \,\mathrm{d}x < +\infty,$$

where the supremum ranges over all finite balls $B \subset \mathbb{R}^d$ and $(f)_B$ is the average value of f over B. Both spaces are complete.

Remark 1.26. Multiple (equivalent) definitions of Hardy spaces exist. Because of continuity with the rest of the Chapter, we have chosen to write $\mathcal{H}^1$ as (a realization) of the homogeneous Triebel-Lizorkin space $\dot F^0_{1,2}$. We refer to [61], Chapter 2, for a thorough presentation of Hardy, BMO and Triebel-Lizorkin spaces.

Remark 1.27. The space BMO may already be seen as a useful substitute for L^∞, as it has the same scaling. However, it is slightly larger $L^\infty/\mathbb{R} \subsetneq \mathrm{BMO}$, as some functions of moderate growth have bounded mean oscillations, such as $\log|x|$ (pp. 140-144 in [100]). On the other hand, BMO functions cannot be too large, as shown by the following inequality: if $B_1 \subset \mathbb{R}^d$ is the unit ball and $\epsilon > 0$, then we have[11]

$$\forall f \in \mathrm{BMO}, \qquad \int \frac{|f(x) - (f)_{B_1}|}{(1+|x|)^d \log(1+|x|)^{2+\epsilon}} \,\mathrm{d}x \le C(d,\epsilon)\|f\|_{\mathrm{BMO}}. \tag{1.13}$$

Proof of inequality (1.13). The proof of this inequality is very close to that of inequality (2) in [100], 1.1.4, p. 141, although we present a slight improvement in order to highlight that BMO functions essentially have logarithmic growth at infinity. For any $k \ge 0$, we note B_k the ball centered in $x = 0$ of radius 2^k. Then,

$$\big|(f)_{B_{k+1}} - (f)_{B_k}\big| \le \big|f(x) - (f)_{B_k}\big| + \big|(f)_{B_{k+1}} - f(x)\big|.$$

By integrating this inequality on B_k and using the definition of the BMO norm, we find that

$$\begin{aligned} 2^{kd}\big|(f)_{B_{k+1}} - (f)_{B_k}\big| &\le 2^{kd}\|f\|_{\mathrm{BMO}} + \int_{B_k} \big|f(x) - (f)_{B_{k+1}}\big| \,\mathrm{d}x \\ &\le \Big(2^{kd} + 2^{(k+1)d}\Big) \|f\|_{\mathrm{BMO}} = 2^{kd}(1+2^d)\|f\|_{\mathrm{BMO}}. \end{aligned}$$

Therefore, by induction, it follows that $|(f)_{B_k} - (f)_{B_1}| \le C(d)k\|f\|_{\mathrm{BMO}}$ for any $k \ge 1$. To prove inequality (1.13), we decompose the integral into dyadic annuli and take advantage of our newfound estimate: by agreeing that $B_0 = \emptyset$, we have

$$\begin{aligned} \int \frac{|f(x) - (f)_{B_1}|}{(1+|x|)^d \log(1+|x|^d)^{2+\epsilon}} \,\mathrm{d}x &= \sum_{k\ge 0} \int_{B_{k+1}\setminus B_k} \frac{|f(x) - (f)_{B_1}|}{(1+|x|)^d \log(1+|x|^d)^{2+\epsilon}} \,\mathrm{d}x \\ &\le \sum_{k\ge 0} \frac{1}{2^{kd}(k+1)^{2+\epsilon}} \int_{B_{k+1}} |f(x) - (f)_{B_1}| \,\mathrm{d}x \\ &\le C(d) \sum_{k\ge 0} \frac{1}{k^{1+\epsilon}} \|f\|_{\mathrm{BMO}}. \end{aligned}$$

[11] In fact, with the same proof, it is possible to state a more precise inequality: by noting the convergence of the series $\sum k^{-1}\log(k)^{-1-\epsilon}$, we see the logarithmic factor can be replaced by $\log(1+|x|)^2 \log(1+\log(1+|x|)^{1+\epsilon})$.

This last inequality ends proving (1.13). □

The Hardy space $\mathcal{H}^1$ is smaller than L^1 (see 2.3.3 p. 112 in [100]). In fact, it is included in the space of functions f of average value zero $\int f = 0$ (5.4 p. 128 in [100]). With respect to Besov spaces, $\mathcal{H}^1$ is in the middle of the chain of embeddings (see 2.2.2 p. 105 in [61])

$$\dot{B}^0_{1,1} \xrightarrow{\subset} \mathcal{H}^1 \xrightarrow{\subset} \dot{B}^0_{1,2}.$$

One of the most useful properties of the Hardy space $\mathcal{H}^1$ is that singular integral operators act continuously on it (paragraph 3.1 pp. 113-114 in [100]). Hence T is bounded $T : \mathcal{H}^1 \longrightarrow \mathcal{H}^1$. This result can be extended to BMO by using duality properties due to Fefferman.

Theorem 1.28 (Theorem 1 p. 142-144 in [100])**.** *Let $\mathcal{D}_0$ be the space of $f \in \mathcal{D}$ such that $\int f = 0$. Then, for any $g \in \mathrm{BMO}$, the linear map defined by*

$$\forall f \in \mathcal{D}_0, \qquad S_g(f) := \int g(x)f(x)\,\mathrm{d}x$$

has a unique bounded extension to $\mathcal{H}^1$. Conversely, any $S \in (\mathcal{H}^1)'$ can be represented as the unique extension of a S_g for some $g \in \mathrm{BMO}$.

Remark 1.29. Note that the integral defining $S_g(f)$ does not depend on the representative of g modulo constant functions, as f is assumed to have no average value $\int f = 0$.

The duality $(\mathcal{H}^1)' = \mathrm{BMO}$ allows us to define the operator T on BMO functions: for any $f \in \mathrm{BMO}$, define Tf by the bracket

$$\forall g \in \mathcal{H}^1, \qquad \langle Tf, g\rangle_{\mathrm{BMO}\times\mathcal{H}^1} := \langle f, T^\dagger g\rangle_{\mathrm{BMO}\times\mathcal{H}^1}.$$

In the above, $T^\dagger$ is the formal transpose of T, whose symbol is $\overline{m(\xi)}$. The continuity of $T : \mathcal{H}^1 \longrightarrow \mathcal{H}^1$ shows that we have defined a bounded operator $T : \mathrm{BMO} \longrightarrow \mathrm{BMO}$.

Just as the Hardy space $\mathcal{H}^1$, the space BMO may be placed in a chain of embeddings involving Besov spaces. The dual embeddings of $\dot{B}^0_{1,1} \hookrightarrow \mathcal{H}^1 \hookrightarrow \dot{B}^0_{1,2}$ above are

$$\dot{B}^0_{\infty,2} \xrightarrow{\subset} \mathrm{BMO} \xrightarrow{\subset} \dot{B}^0_{\infty,\infty}.$$

This shows that considering T as an operator $T : L^\infty \longrightarrow L^\infty/\,\mathbb{R} \longrightarrow \mathrm{BMO}$ is indeed an improvement on what we had before, where we had seen the range of T to be in $\dot{B}^0_{\infty,\infty}$.

Finally, we conclude this paragraph by defining the space of derivatives of BMO functions. Historically, this space was introduced by as a set of possible initial data for solving the Navier-Stokes equations [69].

Definition 1.30 (see Koch and Tataru, [69])**.** Define BMO^{-1} as the set of $f \in \mathcal{S}'$ such that there are functions $g_1, ..., g_d \in \mathrm{BMO}$ with $f = \sum_k \partial_k g_k$.

Elements of BMO^{-1} are distributions, and not classes of distributions modulo constants: the divergence $\sum_k \partial_k g_k$ is defined independently from the representatives of the g_k in BMO. More precisely, we have the following result.

Proposition 1.31. *The space BMO^{-1} lies in $\mathfrak{B}^{-1}_{\infty,\infty}$, the realization of $\dot{B}^{-1}_{\infty,\infty}$ as a subspace of $\mathcal{S}'_h$.*

Proof. From the embedding $\mathrm{BMO} \hookrightarrow \dot{B}^0_{\infty,\infty}$, it is clear that we do indeed have a bounded map $\mathrm{BMO}^{-1} \longrightarrow \dot{B}^{-1}_{\infty,\infty}$. In addition, inequality (1.13) shows that the space BMO contains no polynomial of order 1, so that this map is in fact an embedding $\mathrm{BMO}^{-1} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$.

Since the subcritical Besov space $\dot{B}^{-1}_{\infty,\infty}$ can be realized as a subspace $\mathfrak{B}^{-1}_{\infty,\infty}$ of $\mathcal{S}'_h$, the elements of BMO^{-1} are $\mathcal{S}'_h$ functions up to a polynomial. But neither BMO^{-1} nor $\mathcal{S}'_h$ contain any polynomial, so the Proposition is proved. □

1.7 Chemin's Space of Homogeneous Distributions

Now, it is time to leave the part of this chapter made of tools for the rest of the dissertation and make way for a new Section which concentrates on homogeneous Besov spaces studied for themselves. In all that follows, we will work with *supercritical* spaces: the indices (s, p, r) will satisfy

$$s > \frac{d}{p} \qquad \text{or} \qquad s = \frac{d}{p} \text{ and } r > 1. \tag{1.14}$$

More precisely, we will investigate the role of Chemin's space $\mathcal{S}'_h$ of homogeneous distributions (Definition 1.3) in the structure of homogeneous Besov spaces. As we have explained in Section 1.4 above, subcritical Besov spaces $\dot{B}^s_{p,r}$ (that is with (s, p, r) satisfying $s < d/p$, or $s = d/p$ and $r = 1$) may be realized as subspaces of $\mathcal{S}'$: all representatives modulo $\mathbb{R}[X]$ of functions in $\dot{B}^s_{p,r}$ can be chosen in the *Banach space*

$$\dot{\mathfrak{B}}^s_{p,r} = \left\{ f \in \mathcal{S}'_h, \quad \|f\|_{\dot{B}^s_{p,r}} < +\infty \right\}.$$

In the above, the purpose of the space $\mathcal{S}'_h$ is that all functions in $\mathfrak{B}^s_{p,r}$ are given by their homogeneous Littlewood-Paley decomposition (1.2). In other words, by using the embedding $\mathcal{S}'_h \hookrightarrow \mathcal{S}'/\mathbb{R}[X]$ in order to introduce a slight abuse of notation[12], we have $\dot{B}^s_{p,r} \subset \mathcal{S}'_h$.

On the other hand, although it is common knowledge that $\mathcal{S}'_h \cap \dot{B}^s_{p,r} \subsetneq \dot{B}^s_{p,r}$ if the space $\dot{B}^s_{p,r}$ is supercritical (1.14), little has been said about this strict inclusion. Our goal, in this Section, is to find out how far this inclusion actually is from being an equality.

More precisely, we will prove three properties, which hold as soon as (1.14) does not:

(i) the subspace $\mathcal{S}'_h \cap \dot{B}^s_{p,r}$ is not closed in $\dot{B}^s_{p,r}$; moreover, it is dense in $\dot{B}^s_{p,r}$ if and only if $r < +\infty$ (Theorem 1.37);

(ii) when $r = +\infty$, note $C^s_p = \mathrm{clos}(\mathcal{S}'_h \cap \dot{B}^s_{p,\infty})$ the closure of the intersection in the $\dot{B}^s_{p,\infty}$ topology; then the quotient space $\dot{B}^s_{p,\infty}/C^s_p$ is not separable (Theorem 1.40);

(iii) if $p < +\infty$, the space C^s_p is not complemented in $\dot{B}^s_{p,\infty}$. In other words, there is no decomposition $\dot{B}^s_{p,\infty} = C^s_p \oplus G$ with continuous projections (Theorem 1.41).

Let us comment a bit on these statements. While for $r < +\infty$ there seems not to be much "in between" $\mathcal{S}'_h \cap \dot{B}^s_{p,r}$ and $\dot{B}^s_{p,r}$, it is not so for $\dot{B}^s_{p,\infty}$. The fact that $\dot{B}^s_{p,\infty}/C^s_p$ is not separable means that the inclusion $\mathcal{S}'_h \cap \dot{B}^s_{p,\infty} \subsetneq \dot{B}^s_{p,\infty}$ is indeed very far from being an equality.

The third result regarding the non-complementation of C^s_p must be seen in the same way. As we will see, the proof relies on the construction of an uncountable family of relatively "independent" subspaces of $\dot{B}^s_{p,\infty}/C^s_p$, and is in fact a direct prolongation of the proof we give of (ii).

The study of complementation in Banach spaces is by no means new in the landscape of functional analysis, and has been marked by a number of very deep results (we refer to [24] and the references therein for an enlightening introduction to the topic). Amongst these is the Phillips-Sobczyk theorem [90], [99], which states that the space c_0 of sequences converging to zero is not complemented in the space ℓ^∞ of bounded sequences: there is no bounded projection $P : \ell^\infty \longrightarrow c_0$ (see also [1] Section 2.5 pp. 44-48).

Our result regarding the non-complementation of C^s_p in $\dot{B}^s_{p,\infty}$ is an adaptation of a proof of the Phillips-Sobczyk theorem by Whitley [102], which is based on a countability argument. As

[12] In the rest of this chapter, and as we do here, we will continue to identify $\mathcal{S}'_h$ with its image in $\mathcal{S}'/\mathbb{R}[X]$. Therefore, $\mathcal{S}'_h \cap \dot{B}^s_{p,r}$ is a subspace of $\mathcal{S}'/\mathbb{R}[X]$.

we will see, a well chosen embedding $J : \ell^\infty \longrightarrow \dot{B}^s_{p,\infty}$ will allow the main ingredients of Whitley's proof to find their counterpart in the framework of $\dot{B}^s_{p,\infty}$.

We do certainly not presume to bring any meaningful contribution to the topic of complementation in Banach spaces: our goal is merely to use this theory to illustrate the role of $\mathcal{S}'_h$ in homogeneous Besov spaces.

Let us give a short overview of this Section. We start by discussing general properties and alternative definitions of the space $\mathcal{S}'_h$ in Subsection 1.7.1 before investigating points (i), (ii) and (iii) above in Subsections 1.7.2, 1.7.3 and 1.7.4.

Finally, we end this Section with an investigation of a critical case: as seen in (1.8), the space L^∞ plays a somewhat critical role in this problem, being intermediate between the spaces $\dot{B}^0_{\infty,1}$ and $\dot{B}^0_{\infty,\infty}$ which behave very differently with respect to our problem: the first one is included in $\mathcal{S}'_h$ whereas the second one falls into points (ii) and (iii) above with $p = +\infty$. In Subsection 1.7.5, we try to replace $\dot{B}^s_{p,r}$ and $\dot{B}^s_{p,\infty}$ by L^∞ in points (i), (ii) and (iii) above.

1.7.1 General Remarks on $\mathcal{S}'_h$

When we introduced the space $\mathcal{S}'_h$ in Definition 1.3, we noted that many different definitions coexist. This paragraph is aimed at discussing the various definitions and the way they interact. We give four of them:

(i) We had defined $\mathcal{S}'_h$ as being the space of tempered distributions $f \in \mathcal{S}'$ that fulfill a low-frequency condition

$$\chi(\lambda D)f \underset{\lambda\to+\infty}{\longrightarrow} 0 \qquad \text{in } \mathcal{S}'. \tag{1.15}$$

This definition is particularly interesting with regards to homogeneous Littlewood-Paley theory: the space $\mathcal{S}'_h$ is closely related to[13] the set of distributions such that the series $\sum_{j\in\mathbb{Z}} \dot{\Delta}_j f$ converges to f in $\mathcal{S}'$.

(ii) Alternatively, one may require the convergence above to take place in a stronger topology: as in [7], one could define $\mathcal{S}'_h$ as being the set of $f \in \mathcal{S}'$ such that

$$\chi(\lambda D)f \underset{\lambda\to+\infty}{\longrightarrow} 0 \qquad \text{in } L^\infty. \tag{1.16}$$

The purpose of this definition is mainly to capture realizations of subcritical homogeneous Besov spaces: if the space $\dot{B}^s_{p,r}$ is subcritical, that is if (s,p,r) satisfies (1.9), then, for all $f \in \dot{B}^s_{p,r}$, the series $\sum_{j\leq 0} \dot{\Delta} f$ converges in L^∞ and the function $\sum_{j\in\mathbb{Z}} \dot{\Delta}_j f = f \pmod{\mathbb{R}[X]}$ satisfies (1.16).

(iii) Bourdaud [13] introduced the set of distributions that tend to zero at infinity: that is all the $f \in \mathcal{D}'$ such that $f(\lambda x) \longrightarrow 0$ as $\lambda \to +\infty$ and in the $\mathcal{D}'$ topology. In other words, for all $\phi \in \mathcal{D}$,

$$\left\langle f(x), \frac{1}{\lambda^d}\phi\left(\frac{x}{\lambda}\right)\right\rangle_{\mathcal{D}'\times\mathcal{D}} \longrightarrow 0 \qquad \text{as } \lambda \to +\infty.$$

The intuitive meaning of this convergence is that f has no "average value". For instance, any compactly supported distribution tends to zero at infinity. In [13], it is shown that $\dot{B}^0_{\infty,1}$ can be realized as a space of distributions tending to zero at infinity.

[13]Strictly speaking, the convergence of the Littlewood-Paley decomposition is only equivalent to the weaker condition $\chi(2^{-j}D)f \longrightarrow 0$ as $j \to -\infty$. Convergence to zero of the subsequence does not imply the full convergence when $\lambda \to +\infty$. But this is really a technicality.

(iv) Finally, we may impose on a $f \in \mathcal{S}'$ a condition using the heat kernel:

$$e^{t\Delta} f \underset{t\to+\infty}{\longrightarrow} 0 \qquad \text{in } \mathcal{S}'. \tag{1.17}$$

This condition, although phrased slightly differently, is very similar to (1.15), the difference being that the heat kernel is not spectrally supported in a compact set. The convergence (1.17) aims at eliminating all harmonic components from a given tempered distribution.

Example 1.32. Let us give a few examples. First of all, any tempered distribution whose Fourier transform is integrable in a neighborhood of $\xi = 0$ is in $\mathcal{S}'_h$ according to (ii), which is the strongest of the definitions above: let $f \in \mathcal{S}'$ be such a distribution, because $\chi(\lambda\xi)$ is supported in a ball of radius $O(\lambda^{-1})$, we have

$$\chi(\lambda\xi)\widehat{f}(\xi) \underset{\lambda\to+\infty}{\longrightarrow} 0 \qquad \text{in } L^1,$$

so that $\chi(\lambda D)f$ converges to zero in L^∞. For example, any trigonometric polynomial with zero average value lies in $\mathcal{S}'_h$ according to definition (ii).

Example 1.33. Next, consider the space C_0 of continuous functions that tend to zero at $|x| \to +\infty$. Then $C_0 \subset \mathcal{S}'_h$, again according to definition (ii): let $f \in \mathcal{C}_0$ and $\epsilon > 0$, we may fix a $R > 0$ such that $|f(x)| \leq \epsilon$ for all $|x| \geq R$. Therefore f is equal to the sum $f = g + h$ of a compactly supported function g and function h whose L^∞ norm is smaller than $\|h\|_{L^\infty} \leq \epsilon$, and so, by Example 1.32 above,

$$\|\chi(\lambda D)f(x)\|_{L^\infty} = \|\chi(\lambda D)h(x)\|_{L^\infty} + o(1) \lesssim \epsilon + o(1).$$

Now, since $C_0 \subset \mathcal{S}'_h$ in the sense of (ii), it is also true in the sense of (i).

Example 1.34. In contrast with the two above, another (more subtle) example may help us point out what differences exist between the different definitions above. Let $\sigma = \mathbb{1}_{\mathbb{R}_+} - \mathbb{1}_{\mathbb{R}_-}$ be the sign function. Then we have

$$\chi(\lambda D)\sigma(x) = \int_{-\infty}^{+\infty} \sigma(x-y)\psi_\lambda(y)\mathrm{d}y.$$

Here and in the sequel, we use the following notation: for any function g, and $\lambda > 0$ let $g_\lambda(x) = \lambda^d g(\lambda x)$. This form of $\chi(\lambda D)\sigma$ shows that the sign function cannot possibly satisfy (1.16), as dominated convergence provides[14] the limit $\psi_\lambda * \sigma(x) \longrightarrow \pm 1$ as $x \to \pm\infty$, so $\|\chi(\lambda D)\sigma\|_{L^\infty} = 1$. On the other hand, $\psi_\lambda * \sigma$ tends to zero uniformly locally[15], and so it does in $\mathcal{S}'$. The same argument applies to show that $e^{t\Delta}\sigma \longrightarrow 0$ as $t \to +\infty$ (in $\mathcal{S}'$). Finally, since σ is an odd function and ψ_λ an even one, we have

$$\big\langle \sigma(x), \psi_\lambda(-x) \big\rangle_{L^\infty \times L^1} = 0.$$

However, it must be noted that despite the previous cancellation, the sign function σ does not tend to zero at infinity in the sense of (iii). Taking a nonzero test function $\phi \in \mathcal{D}$ such that $\phi \leq 0$ in $\mathbb{R}_-$ and $\phi \geq 0$ in $\mathbb{R}_+$ gives

$$\int \sigma(\lambda x)\phi(x)\,\mathrm{d}x = \int |\phi| \neq 0.$$

We will study the way definitions (i) and (iii) interact in the special case of $L^\infty(\mathbb{R}^d)$ with $d \geq 1$.

Proposition 1.35. *Consider $f \in L^\infty$. Consider $\psi \in \mathcal{S}$ such that $\widehat{\psi} = \chi$ and define, for $\lambda > 0$, the function $\psi_\lambda(x) = \lambda^{-d}\psi(\lambda^{-1}x)$. The following assertions are equivalent:*

[14] Recall that $\int \psi = \widehat{\psi}(0) = 1$.

[15] In other words, for every compact $K \subset \mathbb{R}^d$, the functions $(\psi_\lambda * \sigma)\mathbb{1}_K$ tend to zero in L^∞.

(i) *we have* $\chi(\lambda D)f \overset{*}{\rightharpoonup} 0$ *in* L^∞, *as* $\lambda \to +\infty$;

(ii) *we have* $\langle f, \psi_\lambda \rangle_{L^\infty \times L^1} \longrightarrow 0$ *as* $\lambda \to +\infty$.

Proof. The link between the two statements is this: the brackets of *(ii)* can be seen as a particular value of a convolution product, namely (recall that χ and ψ are radial functions)

$$\psi_\lambda * f(0) = \left\langle f(y), \frac{1}{\lambda^d}\psi\left(\frac{-y}{\lambda}\right)\right\rangle_{L^\infty \times L^1}.$$

All we have to do is show that the value of $\psi_\lambda * f(x)$ cannot be too far from $\psi_\lambda * f(0)$. This is a consequence of the regularizing nature of ψ_λ. Since $\chi(\lambda D)$ is a low frequency cut-off, the function $\chi(\lambda D)f = \psi_\lambda * f$ is smooth (analytic in fact) with good estimates on its derivatives. Thus, a Taylor expansion is an appropriate way to study the difference between the value at x and at zero of the convolution product:

$$\begin{aligned}\psi_\lambda * f(x) &= \psi_\lambda * f(0) + x \cdot \int_0^1 \nabla\psi_\lambda * f(tx)\mathrm{d}t = \psi_\lambda * f(0) + \frac{x}{\lambda} \cdot \int_0^1 (\nabla\psi)_\lambda * f(tx)\mathrm{d}t \\ &= \psi_\lambda * f(0) + O\left(\frac{|x|}{\lambda}\right),\end{aligned}$$

where the constant in the $O(\,.\,)$ is $\|\nabla\psi\|_{L^1}\|f\|_{L^\infty}$. On the one hand, if *(ii)* holds, then the previous equation shows that $\psi_\lambda * f$ converges locally uniformly to zero, thus giving *(i)*. On the other, by fixing $\phi \in \mathcal{S}$, we obtain

$$\left| \langle \psi_\lambda * f, \phi \rangle_{L^\infty \times L^1} - \psi_\lambda * f(0) \int \phi \right| \leq \frac{1}{\lambda}\|\nabla\psi\|_{L^1}\|f\|_{L^\infty} \int |x||\phi(x)|\,\mathrm{d}x = O\left(\frac{1}{\lambda}\right),$$

so that the convergence of $\psi_\lambda * f(0)$ implies weak convergence of $\chi(\lambda D)f$ to zero. □

Remark 1.36. Proposition 1.35 hints as to which ones of the definitions (i)-(iv) above are relatively stronger. For $f \in L^\infty$, then (ii) implies (iii), which implies both (i) and (iv).

1.7.2 Closure of $\mathcal{S}'_h$ in Besov Spaces

In this paragraph, we focus on the topological properties of the intersection $\mathcal{S}'_h \cap \dot{B}^s_{p,r}$. The following Proposition seems to be common knowledge (especially point *(i)*), although we have been unable to locate a proof in the literature.

We point out that assertion *(ii)* in Proposition 1.37 below finds a close counterpart in Proposition 2.27 and Remark 2.28 in [7]. However, the authors of [7] use a different definition of $\mathcal{S}'_h$.

Proposition 1.37. *Consider* $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$ *such that* $\dot{B}^s_{p,r}$ *is supercritical* $s > d/p$, *or* $s = d/p$ *and* $r > 1$, *then the following assertions hold:*

(i) *the subspace* $\mathcal{S}'_h \cap \dot{B}^s_{p,r}$ *is not closed in* $\dot{B}^s_{p,r}$;

(ii) *the intersection* $\mathcal{S}'_h \cap \dot{B}^s_{p,r}$ *is dense in* $\dot{B}^s_{p,r}$ *if and only if* $r < +\infty$.

Remark 1.38. In particular, with the above choice of exponents (s, p, r), the space $\dot{B}^s_{p,r}$ cannot be realized as a subspace of $\mathcal{S}'_h$. In other words, there is no linear map $\sigma : \dot{B}^s_{p,r} \longrightarrow E$ to a subspace $E \subset \mathcal{S}'$ such that $E \subset \mathcal{S}'_h$ and the following diagram commutes:

$$\begin{tikzcd} \dot{B}^s_{p,r} \arrow[r, "\subset"] \arrow[d, "\sigma"'] & \mathcal{S}'/\mathbb{R}[X] \\ E \arrow[ur, "\pi"'] & \end{tikzcd}$$

where in the above $\pi : E \subset \mathcal{S}' \longrightarrow \mathcal{S}'/\mathbb{R}[X]$ is the natural projection. Indeed, if that were the case, any function $f \in \dot{B}^s_{p,r}$ with $f \notin \mathcal{S}'_h \cap \dot{B}^s_{p,r}$ would define an element $\sigma(f) \in \mathcal{S}'_h$ which would be mapped to an element of $\mathcal{S}'_h \cap \dot{B}^s_{p,r}$ by q, this being a contradiction.

Proof (of Proposition 1.37). We start by showing point *(i)*. The idea of the proof is to exhibit an element $\dot{B}^s_{p,r}$ which does not belong to $\mathcal{S}'_h$ (that is to its image in $\mathcal{S}'/\mathbb{R}[X]$). Let $\psi \in \mathcal{S}$ be a nonzero function with nonnegative Fourier transform $\widehat{\psi} \geq 0$ such that $\varphi(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi)$, where φ is the Littlewood-Paley decomposition function defined in Section 1.3, and define $\psi_j(x) = 2^{jd}\psi(2^j x)$ for $j \in \mathbb{Z}$ so that

$$\dot{\Delta}_j \psi_j = \psi_j \qquad \text{and} \qquad \|\psi_j\|_{L^p} = 2^{jd\left(1-\frac{1}{p}\right)}\|\psi\|_{L^p} \tag{1.18}$$

for all $p \in [1, +\infty]$. We define our function by

$$g = \sum_{-\infty}^{-1} 2^{-jd\left(1-\frac{1}{p}\right)} 2^{-js} \frac{1}{|j|^\alpha}\psi_j, \tag{1.19}$$

where $\alpha \in [1, +\infty]$ is chosen so that

$$\alpha = 2 \text{ if } s > \frac{d}{p} \qquad \text{and} \qquad \alpha = 1 \text{ if } s = \frac{d}{p}.$$

Now, the Besov norm of g is finite: we compute

$$\begin{aligned}\|g\|_{\dot{B}^s_{p,r}} &\approx \left(\sum_{-\infty}^{-1} \left(2^{-jd\left(1-\frac{1}{p}\right)} \|\psi_j\|_{L^p} \right)^r \frac{1}{|j|^{\alpha r}} \right)^{1/r} \\ &\approx \left(\sum_{-\infty}^{-1} \frac{1}{|j|^{\alpha r}} \right)^{1/r},\end{aligned}$$

with the usual modification taking the $\ell^\infty(\mathbb{Z})$ norm if $r = +\infty$. By remembering that $r > 1$ if $s = d/p$, we see that this last sum is finite. In particular, we see that the series (1.19) defining g converges in $\dot{B}^s_{p,r}$, which is a Banach space. Therefore (1.19) does indeed define an element of $\dot{B}^s_{p,r}$.

We must now prove that $g \notin \mathcal{S}'_h \cap \dot{B}^s_{p,r}$, or in other words that the series defining g does not converge in the $\mathcal{S}'$ topology. For this, we fix a $\phi \in \mathcal{S}$ such that $\widehat{\phi}(\xi) \equiv 1$ around $\xi = 0$ and we compute the partial sum

$$\begin{aligned}\left\langle \sum_{-N}^{-1} 2^{j-d\left(1-\frac{1}{p}\right)} 2^{-js} \frac{1}{|j|^\alpha}\psi_j, \phi \right\rangle_{\mathcal{S}'\times\mathcal{S}} &= \sum_{-N}^{-1} 2^{j(d/p-s)} \frac{1}{|j|^\alpha} \int \widehat{\psi}(2^{-j}\xi) 2^{-jd} \mathrm{d}\xi \\ &\approx \sum_{-N}^{-1} 2^{j(d/p-s)} \frac{1}{|j|^\alpha}.\end{aligned}$$

By choice of s and α, this last sum diverges as $N \to +\infty$, and therefore g cannot be an element of $\mathcal{S}'_h \cap \dot{B}^s_{p,r}$. However, since the series (1.19) is convergent in $\dot{B}^s_{p,r}$, the function g is in the closure of $\mathcal{S}'_h \cap \dot{B}^s_{p,r}$, and so the intersection is not closed.

We now get to point *(ii)* of the Proposition. We focus on the case $r < +\infty$, since the case of $r = +\infty$ will be an immediate consequence of Theorem 1.40 below (whose proof is entirely independent). It is simply a matter of noting that for any $f \in \dot{B}^s_{p,r}$, the series

$$f_N := \sum_{-N}^{\infty} \dot{\Delta}_j f$$

lies in $\mathcal{S}'_h \cap \dot{B}^s_{p,r}$ and converges to f in $\dot{B}^s_{p,r}$. □

1.7.3 Non-Separability of the Quotient: the Case of Besov Spaces

Given Proposition 1.37 above, we see that the inclusion $\mathcal{S}'_h \cap \dot{B}^s_{p,r} \subsetneq \dot{B}^s_{p,r}$ is very nearly an equality if $r < +\infty$, but when $r = +\infty$ we have not yet examined the difference between $\mathcal{S}'_h \cap \dot{B}^s_{p,\infty}$ and $\dot{B}^s_{p,\infty}$. We make the following definition.

Definition 1.39. For any $p \in [1, +\infty]$ and $s \geq d/p$, we define C^s_p to be the closure of $\mathcal{S}'_h \cap \dot{B}^s_{p,\infty}$ in the $\dot{B}^s_{p,\infty}$ topology.

At this point, we must note the importance the precise definition of $\mathcal{S}'_h$ plays in our study. When $\mathcal{S}'_h$ is defined in terms of the norm topology of L^∞, as in point (ii) at the beginning of Subsection 1.7.1, then C^s_p is the set of $f \in \dot{B}^s_{p,\infty}$ such that

$$2^{js}\dot{\Delta}_j f \underset{j\to-\infty}{\longrightarrow} 0 \qquad \text{in } L^\infty,$$

as explained in Remark 2.28 of [7]. Things are not the same in our framework (see Definition 1.3 above), as is shown by Example 1.34: the sign function defines an element of $\mathcal{S}'_h \cap \dot{B}^0_{\infty,\infty} \subset C^0_\infty$ for which the previous convergence does not hold.

The next Theorem states that the inclusion $C^s_p \subsetneq \dot{B}^s_{p,\infty}$ is strict. In fact, Theorem 1.40 does more than that, as it expresses the strict inclusion in terms of the size of the quotient $\dot{B}^s_{p,\infty}/C^s_p$, which is not separable.

Theorem 1.40. *Let $p \in [1, +\infty]$ and $s \geq d/p$. The quotient space $\dot{B}^s_{p,\infty}/C^s_p$ is not separable. In fact, there exists an embedding $J : \ell^\infty \longrightarrow \dot{B}^s_{p,\infty}$ which defines a quasi-isometry between the (non-separable) quotient spaces $J' : \ell^\infty/c_0 \longrightarrow \dot{B}^s_{p,\infty}/C^s_p$.*

Proof. We start by defining an embedding $J : \ell^\infty \longrightarrow \dot{B}^s_{p,\infty}$. For any $u \in \ell^\infty$, we formally define

$$Ju = \sum_{-\infty}^{0} u(-j)2^{-jd\left(1-\frac{1}{p}\right)}2^{-js}\psi_j, \tag{1.20}$$

where the functions ψ_j are as in (1.18) above. Now, unlike the series in (1.19), it is not at all obvious that Ju defines an element of $\dot{B}^s_{p,\infty}$ because the sum (1.20) need not converge in $\dot{B}^s_{p,\infty}$ if $u \notin c_0$. Instead, to clarify the meaning of (1.20), we fix $\sigma < d/p$ and set[16]

$$g = (-\Delta)^{s-\sigma}Ju = \sum_{-\infty}^{0} u(-j)2^{jd\left(1-\frac{1}{p}\right)}2^{-js}(-\Delta)^{s-\sigma}\psi_j.$$

Though the sum above does not converge in the Besov space $\dot{B}^\sigma_{p,\infty}$ any more than it did in $\dot{B}^s_{p,\infty}$, the *subcriticality* of $\dot{B}^\sigma_{p,\infty} \subset \dot{B}^{\sigma-d/p}_{\infty,\infty}$ implies it must converge in $\mathcal{S}'$, and so defines a distribution $g \in \mathcal{S}'$ with a finite $\dot{B}^\sigma_{p,\infty}$ norm and therefore an element of $\dot{B}^\sigma_{p,\infty}$. Since the fractional Laplacian defines a quasi-isometry (see Proposition 1.15 above)

$$(-\Delta)^{\sigma-s} : \dot{B}^\sigma_{p,\infty} \longrightarrow \dot{B}^s_{p,\infty},$$

we can in turn define J by the formula $Ju := (-\Delta)^{\sigma-s}g$.

We now are ready to define our map $J' : \ell^\infty/c_0 \longrightarrow \dot{B}^s_{p,\infty}/C^s_p$. First of all, we note that if $u \in c_0$ then the series (1.20) is convergent in the $\dot{B}^s_{p,\infty}$ topology, so that Ju is a $\dot{B}^s_{p,\infty}$ limit of

[16] We cannot simply take $\sigma = 0$, it would be insufficient when $p = +\infty$.

functions whose Fourier transform is supported away from $\xi = 0$. We deduce that $J(c_0) \subset C^s_p$. As a consequence, we may define a quotient map J' such that the following diagram commutes:

$$\begin{array}{ccc} \ell^\infty & \xrightarrow{\quad J \quad} & \dot{B}^s_{p,\infty} \\ \downarrow & & \downarrow \\ \ell^\infty/c_0 & \xrightarrow{\quad J' \quad} & \dot{B}^s_{p,\infty}/C^s_p \end{array}$$

where the vertical maps are the natural projections. To conclude, we must show that J' is a quasi-isometry. In order to do so, we will prove that the functions of C^s_p inherit a low-frequency property from $\mathcal{S}'_h \cap \dot{B}^s_{p,\infty}$ which Ju cannot posses if $u \notin c_0$. More precisely, we prove that if $g \in L^1$ and $f \in \dot{B}^s_{p,\infty}$, then we have:

$$\overline{\lim_{j\to-\infty}} \left| \left\langle 2^{j(s-d/p)} \dot{\Delta}_j f, g \right\rangle_{L^\infty \times L^1} \right| \leq \|g\|_{L^1} \mathrm{dist}(f, C^s_p) = \|g\|_{L^1} \inf_{h \in C^s_p} \|f - h\|_{\dot{B}^s_{p,\infty}}. \tag{1.21}$$

Taking inequality (1.21) for granted and leaving its proof for later, we are nearly done: by using (1.18), we see that the dyadic blocks of Ju are

$$2^{j(s-d/p)} \dot{\Delta}_j Ju(x) = u(-j)\psi(2^j x)$$

and so, by dominated convergence,

$$\left\langle 2^{j(s-d/p)} \dot{\Delta}_j Ju, g \right\rangle_{L^\infty \times L^1} \sim u(-j)\, \psi(0) \int g \qquad \text{as } j \to -\infty.$$

Finally, by choosing $g \in \mathcal{S}$ such that $\int g \neq 0$ and noting that, on the one hand $\psi(0) = \int \widehat{\psi} > 0$, and on the other that

$$\overline{\lim_{j\to-\infty}} |u(-j)| \;=\; \inf_{w \in c_0} \|u - w\|_{\ell^\infty} \;=\; \|u\|_{\ell^\infty/c_0}, \tag{1.22}$$

we see that J' is indeed a quasi-isometry. Let us prove this last assertion (1.22). Firstly, it is clear that for all $w \in c_0$, we must have

$$\overline{\lim_{j\to-\infty}} |u(-j)| \;=\; \overline{\lim_{j\to-\infty}} |u(-j) - w(-j)| \leq \inf_{w \in c_0} \|u - w\|_{\ell^\infty}.$$

In order to get the reverse inequality, we use the definition of the limit superior as an infimum of suprema. We have

$$\begin{aligned} \overline{\lim_{j\to-\infty}} |u(-j)| \;&=\; \inf_{J\to-\infty} \sup_{j \leq J} \big| u(-j) \big| \\ &=\; \inf_{J\to-\infty} \|u - \mathbb{1}_{[-J,0]} u\|_{\ell^\infty} \geq \inf_{w \in c_0} \|u - w\|_{\ell^\infty}, \end{aligned}$$

because the sequence $\mathbb{1}_{[-J,0]} u$ is finitely supported, and so must lie in c_0. Both inequalities prove that equation (1.22) holds.

It only remains to prove (1.21). First of all, we remark that if $h \in \mathcal{S}'_h \cap \dot{B}^s_{p,\infty}$ and $g \in \mathcal{S}$, the condition $s \geq d/p$ implies that for all $j \leq 0$,

$$\left| \left\langle 2^{j(s-d/p)} \dot{\Delta}_j h, g \right\rangle_{L^\infty \times L^1} \right| \leq \left| \left\langle \dot{\Delta}_j h, g \right\rangle_{\mathcal{S}' \times \mathcal{S}} \right| \xrightarrow[j\to-\infty]{} 0.$$

Next, if $h \in C_p^s$, we may fix a sequence of functions $h_k \in \mathcal{S}'_h \cap \dot{B}^s_{p,\infty}$ that converge to h in $\dot{B}^s_{p,\infty}$. Then, by using the Bernstein inequalities, we obtain

$$
\begin{aligned}
\left|\left\langle 2^{j(s-d/p)}\dot{\Delta}_j h, g\right\rangle_{L^\infty\times L^1}\right| &\leq 2^{j(s-d/p)}\|\dot{\Delta}_j(h-h_k)\|_{L^\infty}\|g\|_{L^1} \\
&\qquad + \left|\left\langle 2^{j(s-d/p)}\dot{\Delta}_j h_k, g\right\rangle_{L^\infty\times L^1}\right| \\
&\leq \|h-h_k\|_{\dot{B}^s_{p,\infty}}\|g\|_{L^1} + \left|\left\langle 2^{j(s-d/p)}\dot{\Delta}_j h_k, g\right\rangle_{L^\infty\times L^1}\right|
\end{aligned} \tag{1.23}
$$

and so

$$
\varlimsup_{j\to-\infty}\left|\left\langle 2^{j(s-d/p)}\dot{\Delta}_j h, g\right\rangle_{L^\infty\times L^1}\right| \leq \|h-h_k\|_{\dot{B}^s_{p,\infty}}\|g\|_{L^1} \underset{k\to+\infty}{\longrightarrow} 0.
$$

which implies that the lefthand side of this last inequality must be zero. Finally, by proceeding exactly as in (1.23), we see that for all $f \in \dot{B}^s_{p,\infty}$ and all $h \in C_p^s$, we have a similar inequality

$$
\varlimsup_{j\to-\infty}\left|\left\langle 2^{j(s-d/p)}\dot{\Delta}_j f, g\right\rangle_{L^\infty\times L^1}\right| \leq \|f-h\|_{\dot{B}^s_{p,\infty}}\|g\|_{L^1},
$$

which gives in turn (1.21). □

1.7.4 Non-Complementation of the Closure C_p^s

In this paragraph, we reach the core of our study on $\dot{B}^s_{p,\infty}$ and C_p^s. The aim of Theorem 1.41 is to reveal the role of C_p^s in the structure of $\dot{B}^s_{p,\infty}$. Precisely, we show that C_p^s is not complemented in $\dot{B}^s_{p,\infty}$: there is no continuous projection $P : \dot{B}^s_{p,\infty} \longrightarrow \dot{B}^s_{p,\infty}$ with range exactly C_p^s. The spirit of this property is to show how different C_p^s and $\dot{B}^s_{p,\infty}$ really are: the proof of Theorem 1.41 heavily relies on the the fact that $\dot{B}^s_{p,\infty}/C_p^s$ is very large (in fact large enough to contain an isomorphic copy of ℓ^∞/c_0).

Theorem 1.41. *Let $p \in]1,+\infty]$ and $s \geq d/p$. Then C_p^s is not complemented in $\dot{B}^s_{p,\infty}$: there is no decomposition $\dot{B}^s_{p,\infty} = C_p^s \oplus G$ with continuous projections.*

Remark 1.42. In Theorem 1.41, the Lebesgue exponent must be taken $p > 1$. The reason is that when $p = 1$, the Besov spaces $\dot{B}^s_{1,r}$ do not obviously have a separable predual, even when $r > 1$. Whether the non-complementation property remains true in that case is still unknown.

The spirit of what follows is to adapt the ideas of Whitley [102] (see also [1], Section 2.5) in his proof of the Phillips-Sobczyk theorem, which states that c_0 is not complemented in ℓ^∞. Analysis of Whitley's proof, which is based on a countability argument, reveals two key features which we will need to adapt in the framework of Besov spaces:

(i) the existence of an uncountable family of subspaces $\ell^\infty(A_i) \subset \ell^\infty(\mathbb{N})$, for $i \in I$, that are not in c_0 and such that the intersection of any two of these spaces is in c_0; in other words, they are mutually independent up to elements of c_0;

(ii) the fact that the separation of points can be tested by a countable set of equalities: for all $u \in \ell^\infty$, we have $u = 0$ if and only if $u(n) = 0$ for all $n \in \mathbb{N}$.

While both these facts seem very specific to ℓ^∞, we will find homologous assertions in $\dot{B}^s_{p,\infty}$. Firstly, we will see that the embedding $J : \ell^\infty \longrightarrow \dot{B}^s_{p,\infty}$ of Theorem 1.40 preserves the properties of the spaces $\ell^\infty(A_i)$ used in Whitley's argument. Secondly, the space $\dot{B}^s_{p,\infty}$ has a separable[17]

[17]Recall from Theorem 1.17 that we note p' the conjugate Lebesgue exponent $\frac{1}{p} + \frac{1}{p'} = 1$.

predual space $\dot{B}^{-s}_{p',1}$ for all $p > 1$ (see Theorem 1.17 above), and so the separation of points can be tested by a countable number of equalities.

STEP 1. We start by constructing an uncountable family of subspaces of $\dot{B}^s_{p,\infty}$ such that the intersection of any two of these spaces lies in C^s_p. The existence of such spaces will stem from the following Lemma (see for example Lemma 2.5.3 in [1]), which we reproduce and prove for the reader's convenience.

Lemma 1.43. *There exists an uncountable family $(A_i)_{i\in I}$ of infinite subsets of $\mathbb{N}$ such that, for any two $i \neq j$, there is a finite intersection $|A_i \cap A_j| < +\infty$.*

Proof (of the Lemma). Since only countability matters in the statement we wish to prove, nothing is lost in replacing $\mathbb{N}$ by $\mathbb{Q}$ and seeking the A_i as subsets of $\mathbb{Q}$. Next, for any irrational $\theta \in \mathbb{R}\setminus\mathbb{Q}$, fix a sequence (q_k) of rational numbers such that $q_k \to \theta$.

Define $A_\theta = \{q_k,\ k \geq 0\}$. Then the sets $(A_\theta)_{\theta\in\mathbb{R}}$ are all infinite and any two of these sets must have finite intersection. □

In particular, the subspaces $\ell^\infty(A_i)$ of ℓ^∞ sequences which are compactly supported in A_i have the following properties: for $i \neq j$,

$$\ell^\infty(A_i) \not\subset c_0 \qquad \text{and} \qquad \ell^\infty(A_i) \cap \ell^\infty(A_j) \subset c_0.$$

In what follows, we will transport these spaces into $\dot{B}^s_{p,\infty}$ by means of a well chosen map: recall $J : \ell^\infty \longrightarrow \dot{B}^s_{p,\infty}$ from Theorem 1.40, which we have seen to satisfy $J^{-1}(C^s_p) = c_0$ so that $Ju \in C^s_p$ if and only if $u \in c_0$. This implies that the image spaces $J\big(\ell^\infty(A_i)\big)$ satisfy

$$J\big(\ell^\infty(A_i)\big) \not\subset C^s_p \qquad \text{and} \qquad J\big(\ell^\infty(A_i) \cap \ell^\infty(A_j)\big) \subset C^s_p$$

and we see that the spaces $J\big(\ell^\infty(A_i)\big)$ will be well-suited for our purpose.

STEP 2. Consider a bounded operator $T : \dot{B}^s_{p,\infty} \longrightarrow \dot{B}^s_{p,\infty}$ such that $C^s_p \subset \ker(T)$. We will prove the existence of a $i \in I$ such that $J\big(\ell^\infty(A_i)\big) \subset \ker(T)$.

Assume, in order to obtain a contradiction, that none of the $J\big(\ell^\infty(A_i)\big)$ lie in the kernel of T. Therefore, for every $i \in I$, we may find a $u_i \in \ell^\infty(A_i)$ such that $TJu_i \neq 0$. In addition, we may assume u to be in the unit ball $\|u\|_{\ell^\infty} \leq 1$.

Next, because the predual space $\dot{B}^{-s}_{p',1}$ of $\dot{B}^s_{p,\infty}$ is separable (remember that $p > 1$), we may fix a sequence $(g_n)_n$ which forms a dense subset of the unit ball of $\dot{B}^{-s}_{p',1}$. Then

$$\begin{aligned} I = \{i \in I,\ TJu_i \neq 0\} &= \bigcup_{n\geq 0} \left\{ i \in I, \quad \langle TJu_i, g_n\rangle_{\dot{B}^s_{p,\infty}\times\dot{B}^{-s}_{p',1}} \neq 0 \right\} \\ &= \bigcup_{n,k\geq 0} \left\{ i \in I, \quad \left|\langle TJu_i, g_n\rangle_{\dot{B}^s_{p,\infty}\times\dot{B}^{-s}_{p',1}}\right| \geq \frac{1}{k+1} \right\} := \bigcup_{n,k\geq 0} I_{n,k}. \end{aligned}$$

Because I is uncountable and is the countable union of the $I_{n,k}$, there must exist indices $n, k \geq 0$ such that the set $I_{n,k}$ is also uncountable: in particular, there exists an infinite number of $i \in I$ such that the bracket $\langle TJu_i, g_n\rangle_{\dot{B}^s_{p,\infty}\times\dot{B}^{-s}_{p',1}}$ is not small.

To take advantage of this last fact, we will construct a linear combination of the u_i which will make the bracket become arbitrarily large. Fix a finite subset $F \subset I_{k,n}$ and define the sequence

$$y = \sum_{i\in F} \alpha_i u_i,$$

where the α_i are chosen so that the bracket $\langle TJy, g_n\rangle_{\dot{B}^s_{p,\infty}\times\dot{B}^{-s}_{p',1}}$ becomes large:

$$\alpha_i = \frac{\langle TJu_i, g_n\rangle_{\dot{B}^s_{p,\infty}\times\dot{B}^{-s}_{p',1}}}{|\langle TJu_i, g_n\rangle_{\dot{B}^s_{p,\infty}\times\dot{B}^{-s}_{p',1}}|} = \pm 1.$$

Therefore

$$\left|\langle TJy, g_n\rangle_{\dot{B}^s_{p,\infty}\times\dot{B}^{-s}_{p',1}}\right| \geq \frac{|F|}{k+1}, \tag{1.24}$$

and this lower bound may be made as large as desired by taking $|F|$ as large as needed, the set $I_{k,n}$ being uncountably infinite. On the other hand, because the subsets A_i have finite intersection, we may decompose the union of the A_i as $\cup_{i\in F}A_i = A \sqcup B$ where

$$B = \bigcup_{i\neq j}(A_i \cap A_j) \tag{1.25}$$

the union ranging on all $i, j \in F$ such that $i \neq j$, is a finite set and any $m \in A$ is in exactly one of the A_i. By setting $a = \mathbb{1}_A y$ and $b = \mathbb{1}_B y$, we see that $y = a + b$ with $\|a\|_{\ell^\infty} \leq 1$ and b having finite support. Since b has finite support, $Jb \in C^s_p$ and $TJy = TJa$, so

$$\left|\langle TJu_i, g_n\rangle_{\dot{B}^s_{p,\infty}\times\dot{B}^{-s}_{p',1}}\right| \leq \|T\|. \tag{1.26}$$

By comparing (1.24) and (1.26), we have obtained the contradiction we were seeking, since the set F can be chosen as large as desired, $I_{k,n}$ being uncountably infinite.

STEP 3. We may now end the proof of Theorem 1.41.

Proof of Theorem 1.41. Assume on the contrary that C^s_p has a topological supplementary $\dot{B}^s_{p,\infty} = C^s_p \oplus G$ and let $T : \dot{B}^s_{p,\infty} \longrightarrow G$ be the associated projection on G. Then T is a bounded operator such that $C^s_p = \ker(T)$ and step 2 gives a $i \in I$ such that $J\big(\ell^\infty(A_i)\big) \subset C^s_p$. But this is a contradiction: Theorem 1.40 asserts that Ju cannot lie in C^s_p if $u \notin c_0$, which is certainly the case if $u = \mathbb{1}_{A_i} \in \ell^\infty(A_i)$. □

1.7.5 The Critical Case: the Intersection $\mathcal{S}'_h \cap L^\infty$

So far in our study, it appears that the space L^∞ of bounded functions plays a critical role in that it lies at the interface between the two very different behaviors the homogeneous Besov spaces have: L^∞ is in the center of the chain of embeddings

$$\dot{B}^0_{\infty,1} \xhookrightarrow{\subset} L^\infty \twoheadrightarrow L^\infty/\mathbb{R} \xhookrightarrow{\subset} \dot{B}^0_{\infty,\infty},$$

while on the one hand $\dot{B}^0_{\infty,1} \subset \mathcal{S}'_h$, and on the other $\dot{B}^0_{\infty,\infty}$ gathers all the properties described in Theorem 1.40 and 1.41. A very natural question is whether the space $\mathcal{S}'_h \cap L^\infty$ has an analogous role in the structure of L^∞ as the Besov spaces did.

The Intersection is Closed

Our first answer shows a difference between L^∞ and Besov spaces. While $\mathcal{S}'_h \cap \dot{B}^0_{\infty,\infty}$ was not closed in $\dot{B}^0_{\infty,\infty}$, the intersection $\mathcal{S}'_h \cap L^\infty$ is.

Proposition 1.44. *The space $L^\infty \cap \mathcal{S}'_h$ is closed in L^∞ for the strong topology.*

Proof. Let $(f_n)_{n\geq 0}$ be a converging sequence of functions in $L^\infty \cap \mathcal{S}'_h$ whose limit is $f \in L^\infty$. We have, for all $\phi \in \mathcal{S}$,

$$\begin{aligned}\left|\langle\chi(\lambda D)f, \phi\rangle_{L^\infty\times L^1}\right| &\leq \left|\langle\chi(\lambda D)(f-f_n), \phi\rangle_{L^\infty\times L^1}\right| + \left|\langle\chi(\lambda D)f_n, \phi\rangle_{L^\infty\times L^1}\right| \\ &\leq \|\chi(\lambda D)(f-f_n)\|_{L^\infty}\|\phi\|_{L^1} + \left|\langle\chi(\lambda D)f_n, \phi\rangle_{\mathcal{S}'\times\mathcal{S}}\right|.\end{aligned}$$

The fact that the $\chi(\lambda D)f_n$ converge to 0 in $\mathcal{S}'$ as $\lambda \to +\infty$ shows that we have,

$$\forall n \geq 0, \qquad \overline{\lim_{\lambda\to+\infty}}\left|\langle\chi(\lambda D)f, \phi\rangle_{\mathcal{S}'\times\mathcal{S}}\right| \leq C\,\|(f-f_n)\|_{L^\infty}\|\phi\|_{L^1}.$$

This term has limit 0 as $n \to +\infty$ so that f indeed lies in $L^\infty \cap \mathcal{S}'$. □

Non-Separability of the Quotient

Theorem 1.45. *The quotient space $L^\infty/(\mathcal{S}'_h \cap L^\infty)$ is not separable. In fact, there exists an embedding $H : \ell^\infty \longrightarrow L^\infty$ which defines a quasi-isometry between the (non-separable) quotient spaces $H' : \ell^\infty/c_0 \longrightarrow L^\infty/(\mathcal{S}'_h \cap L^\infty)$.*

Proof. To construct the map $H : \ell^\infty \longrightarrow L^\infty$, the idea is to take advantage of Proposition 1.35 which states that the space $\mathcal{S}'_h$ is characterized by convergence of a family of average values: if χ is the Littlewood-Paley decomposition function, as in Section 1.3 and $\psi \in \mathcal{S}$ such that $\widehat{\psi} = \chi$, then a bounded function $f \in L^\infty$ is in $\mathcal{S}'_h$ if and only if there is convergence of the average values

$$\langle f, \psi_\lambda \rangle_{L^\infty \times L^1} = \int f(x)\, \psi\left(\frac{x}{\lambda}\right) \frac{\mathrm{d}x}{\lambda^d} \underset{\lambda \to +\infty}{\longrightarrow} 0.$$

For any $u \in \ell^\infty$, we will construct a function $Hu \in L^\infty$ whose average values $\langle f, \psi_\lambda \rangle_{L^\infty \times L^1}$ will share accumulation points with the sequence u when $\lambda \to +\infty$.

Consider an increasing sequence $(r_m)_m$ of radii (which we will fix later on) with $r_0 = 0$ and define a family of annuli by $\mathcal{C}_m = \{r_m \leq |x| \leq r_{m+1}\}$. For every sequence $u \in \ell^\infty$, we set

$$Hu = \sum_{m=0}^{\infty} u(m) \mathbb{1}_{\mathcal{C}_m},$$

where the sum is to be understood in the sense of pointwise convergence. First of all, the map $H : \ell^\infty \longrightarrow L^\infty$ is bounded. Next, if $u \in c_0$, then Hu has limit zero at $|x| \to +\infty$, and so $Hu \in C_0 \subset \mathcal{S}'_h \cap L^\infty$ (Example 1.33 shows that the space C_0 of continuous function that tend to zero at infinity lies in $\mathcal{S}'_h$). The inclusion $H(c_0) \subset \mathcal{S}'_h \cap L^\infty$ allows us to define a quotient map H' such that the following diagram commutes:

$$\begin{tikzcd} \ell^\infty \arrow[r, "H"] \arrow[d] & L^\infty \arrow[d] \\ \ell^\infty/c_0 \arrow[r, "H'"] & L^\infty/(\mathcal{S}'_h \cap L^\infty) \end{tikzcd}$$

To fall back on the arguments of Theorem 1.40, we must now show the converse: that if $Hu \in \mathcal{S}'_h \cap L^\infty$ then we must have $u \in c_0$. For this, we prove an inequality that is quite analogous to (1.21). We show that

$$\text{(1.27)} \qquad \forall f \in L^\infty, \qquad \overline{\lim_{\lambda \to +\infty}} |\langle f, \psi_\lambda \rangle_{L^\infty \times L^1}| \leq \operatorname{dist}(f, \mathcal{S}'_h \cap L^\infty) := \inf_{h \in \mathcal{S}'_h \cap L^\infty} \|f - h\|_{L^\infty}.$$

The argument is *mutatis mutandi* the same as for (1.21). On the one hand, in light of Proposition 1.35, it is clear that for any $h \in \mathcal{S}'_h \cap L^\infty$ we must have $\langle h, \psi_\lambda \rangle_{L^\infty \times L^1} \longrightarrow 0$. On the other hand, for $f \in L^\infty$ and $h \in \mathcal{S}'_h \cap L^\infty$, we have

$$|\langle f, \psi_\lambda \rangle_{L^\infty \times L^1}| \leq \|f - h\|_{L^\infty} \|\psi\|_{L^1} + |\langle h, \psi_\lambda \rangle_{L^\infty \times L^1}|.$$

By taking the limit $\lambda \to +\infty$, we deduce (1.27).

With (1.27) at our disposal, we may show that $Hu \notin \mathcal{S}'_h \cap L^\infty$ if u does not belong to c_0. We start by fixing a $u \in \ell^\infty$. Consider a $\lambda > 0$ whose value will be decided later, we have

$$\text{(1.28)} \qquad \int \psi_\lambda Hu = \sum_{m=0}^{\infty} u(m) \int_{\mathcal{C}_m} \psi_\lambda = \sum_{m=0}^{\infty} u(m) \int_{\lambda^{-1}\mathcal{C}_m} \psi.$$

Let $\epsilon > 0$ and fix a $R > 0$ such that

$$\|\mathbb{1}_{|x| \geq R} \psi(x)\|_{L^1} = \|\mathbb{1}_{|x| \geq R\lambda} \psi_\lambda(x)\|_{L^1} \leq \epsilon.$$

On the one hand, the terms of (1.28) at high indices m will be small. More precisely,

$$\left|\sum_{r_m\geq\lambda R} u(m)\int_{\lambda^{-1}\mathcal{C}_m}\psi\right| \leq \|u\|_{\ell^\infty}\int_{|x|\geq R}|\psi(x)|\mathrm{d}x \leq \epsilon\|u\|_{\ell^\infty}. \tag{1.29}$$

On the other hand, terms at low indices will also be small if λ is large, because the annuli $\lambda^{-1}\mathcal{C}_m$ have a small measure: fix a $M\geq 0$, then

$$\left|\sum_{m=0}^{M-1} u(m)\int_{\lambda^{-1}\mathcal{C}_m}\psi\right| \leq \|u\|_{\ell^\infty}\|\psi\|_{L^\infty}\frac{1}{\lambda^d}\sum_{m=0}^{M-1}|\mathcal{C}_m| = \|u\|_{\ell^\infty}\|\psi\|_{L^\infty}\left(\frac{r_M}{\lambda}\right)^d. \tag{1.30}$$

We set the values of the r_m, and λ so that both sums (1.29) and (1.30) are negligible and only the contribution of one term matters. Fix a $M\geq 0$ and set

$$r_m = 2^{m^2} \qquad \text{and} \qquad \lambda = r_{M+1}$$

so that $r_M/\lambda = O(4^{-M})$. The sum in (1.29) ranges on all indices m such that $r_m\geq r_{M+1}R$, that is all $m\geq 0$ with

$$2^{m^2}\geq 2^{2M+1}2^{M^2}R.$$

Therefore, any $m\geq M+1$ is included in that sum if M is taken large enough that $2^{2M+1}R>1$. For such M, we may bound the difference between the full sum (1.28) and the M-th term:

$$\left|\int\psi_\lambda Hu - u(M)\int_{\mathcal{C}_M}\psi_\lambda\right| \leq \|u\|_{\ell^\infty}\left(\epsilon+\frac{\|\psi\|_{L^\infty}}{4^M}\right).$$

Finally, the principal term is equal to $u(M)\int\psi$ up to a small remainder: by using exactly the same bound as in (1.29) and (1.30), we see that

$$\left|\int_{{}^c\mathcal{C}_M}\psi_\lambda\right|\leq\epsilon+\frac{\|\psi\|_{L^\infty}}{4^M}.$$

Therefore, by taking small ϵ, large M and $\lambda = r_{M+1}$, we see that we can make the bracket $\langle Hu,\psi_\lambda\rangle_{L^\infty\times L^1}$ arbitrarily close to any $u(M)$, so all the accumulation points of u are also accumulation points of the bracket as $\lambda\to+\infty$. We deduce:

$$\|u\|_{\ell^\infty/c_0} = \varlimsup_{M\to+\infty}|u(M)| \leq \varlimsup_{\lambda\to+\infty}\left|\langle Hu,\psi_\lambda\rangle_{L^\infty\times L^1}\right|,$$

which ends the proof. □

Non-Complementation of the Intersection

In this final Section, we exploit the embedding $H:\ell^\infty/c_0\longrightarrow L^\infty/(\mathcal{S}'_h\cap L^\infty)$ we have constructed in Theorem 1.45 above to show that $\mathcal{S}'_h\cap L^\infty$ is not complemented in L^∞.

Theorem 1.46. *The space $\mathcal{S}'_h\cap L^\infty$ is not complemented in L^∞. There exists no decomposition $L^\infty = (\mathcal{S}'_h\cap L^\infty)\oplus G$ with continuous projections.*

The proof of Theorem 1.46 will be very similar to that of Theorem 1.41. In fact, we may give an abstract general principle which captures the essence of Whitley's argument and "lifts" it to another Banach space X provided it contains a "suitable" copy of ℓ^∞. Theorem 1.46 is a direct consequence of Theorem 1.45 and the following Proposition.

Proposition 1.47. *Let X be a Banach space that has the following property: there exists a countable family $(g_n)_{n\geq 1}$ of bounded linear functionals $g_n \in X'$ such that*

$$\forall f \in X, \qquad \Big(\forall n \geq 1,\ \langle g_n, f\rangle_{X'\times X} = 0\Big) \ \Rightarrow\ f = 0.$$

In particular, if X has a separable predual, then this property is fulfilled. Now, consider a closed subspace $E \subset X$ and assume the existence of a bounded map $J : \ell^\infty \longrightarrow X$ that defines an embedding of the quotient $J' : \ell^\infty/c_0 \longrightarrow X/E$ such that the diagram

$$\begin{tikzcd} \ell^\infty \arrow[r, "J"] \arrow[d] & X \arrow[d] \\ \ell^\infty/c_0 \arrow[r, "J'"] & X/E \end{tikzcd}$$

commutes (the vertical arrows are again the natural projections). Then E is not complemented in X.

Proof of Proposition 1.47. Consider $T : X \longrightarrow X$ a bounded operator such that $E \subset \ker(T)$ and let $(A_i)_{i\in I}$ be the family of subspaces given by Lemma 1.43. We show that there must be a $i \in I$ with $J\big(\ell^\infty(A_i)\big) \subset \ker(T)$. Assume on the contrary that for all $i \in I$ there is a u_i such that $TJu_i \neq 0$. Then, if $(g_n)_n$ is as in the statement of Proposition 1.47,

$$I = \{i \in I,\ TJu_i \neq 0\} = \bigcup_{k,n\geq 0} \left\{ i \in I, \quad |\langle g_n, TJu_i\rangle_{X'\times X}| \geq \frac{1}{k+1} \right\} := \bigcup_{k,n\geq 0} I_{k,n}.$$

Now, since I is uncountable, there must be indices $k, n \geq 0$ such that $I_{k,n}$ is also uncountable. Next, for any finite $F \subset I_{k,n}$, let

$$y = \sum_{i\in F} \alpha_i u_i, \qquad \text{where} \qquad \alpha_i = \frac{\langle g_n, TJu_i\rangle_{X'\times X}}{|\langle Tg_n, Ju_i\rangle_{X'\times X}|}.$$

In particular,

$$|\langle g_n, TJy\rangle_{X'\times X}| \geq \frac{|F|}{k+1}. \tag{1.31}$$

Next, thanks to the properties of the A_i given by Lemma 1.43, we can decompose the union of the A_i in $\cup_{i\in F} A \sqcup B$, where A and B are given exactly as in (1.25). By setting $y = a + b = \mathbb{1}_A y + \mathbb{1}_B y$, we have $TJy = TJa$ and

$$|\langle TJy, g_n\rangle_{X'\times X}| \leq \|T\|,$$

which contradicts (1.31), since the set F can be chosen as large as desired. $\square$

Remark 1.48. Of course, the abstract principle of Proposition 1.47 is in reality a small part of the proof, the core of the argument is the construction of the map J'. Nevertheless, the same arguments may be used in a number of different frameworks. Let us give an example of a seemingly very different situation where this principle works.

Let H be a real separable Hilbert space and $X := \mathcal{L}(H)$ the space of bounded linear operators on H. Define the subspace $E = \mathcal{K}(H)$ of compact operators. N.J. Kalton showed (see Theorem 6 in [66]) that $\mathcal{K}(H)$ is uncomplemented in $\mathcal{L}(H)$. The argument, presented in a simpler form[18] in

[18] Kalton's result is much more general and expresses the complementation of the space $\mathcal{K}(X,Y)$ of compact operators between two Banach spaces in terms of whether $\mathcal{L}(X,Y)$ contains or not a copy of ℓ^∞. The proof in [24] is adapted to the simpler framework of Hilbert spaces.

[24] (Theorem 6.1, pp. 351-354), although phrased differently, can be reformulated to fit in our framework. Define, for any $u \in \ell^\infty$, the operator $Ju : H \longrightarrow H$ by

$$Ju(x) = \sum_{n=1}^{\infty} u(n) \langle e_n, x \rangle_H \, e_n,$$

where $(e_n)_{n \geq 1}$ is a Hilbert basis of H. Then Ju is compact if and only if $u \in c_0$, as it is in that case a limit of finite rank operators, so we get an embedding $J' : \ell^\infty / c_0 \longrightarrow \mathcal{L}(H)/\mathcal{K}(H)$. In addition, any $T \in \mathcal{L}(H)$ is equal to $T = 0$ if and only if

$$\forall n, m \geq 1, \qquad g_{n,m}(T) := \langle e_n, T e_m \rangle_H = 0,$$

so that there is a countable number of bounded linear maps $g_{n,m} : \mathcal{L}(H) \longrightarrow \mathbb{R}$ that fulfill the assumptions of Proposition 1.47. Our abstract principle (Proposition 1.47) therefore applies to show that $\mathcal{K}(H)$ is uncomplemented in $\mathcal{L}(H)$.

Chapter 2

Bounded Solutions in Incompressible Hydrodynamics

Si on reconnaît qu'il y a une vérité,
il n'est permis de penser que ce qui est vrai.
Simone Weil[1]

2.1 Introduction

In this chapter, we study some specific features of incompressible fluids evolving in unbounded domains. Although the methods developed here will be put to use for the magnetohydrodynamic equations in the next chapter, they are interesting in themselves and will be studied in the case of the incompressible Euler system, which concentrates the difficulty of more complex equations. The material in this chapter is adapted from our own research [25].

We are concerned with *bounded* solutions to systems of PDEs describing incompressible fluid mechanics. The simplest of these equations is the incompressible (homogeneous) Euler system, which reads

$$\begin{cases} \partial_t u + \mathrm{div}(u \otimes u) + \nabla \pi = 0 \\ \mathrm{div}(u) = 0. \end{cases} \tag{2.1}$$

This system is set on the whole space $\mathbb{R}^d$, and its unknowns are the velocity field $u(t, x) \in \mathbb{R}^d$ and the pressure field $\pi(t, x) \in \mathbb{R}$. As we will see, the issues raised in this article are very specific to the non-compact nature of $\mathbb{R}^d$, so we will not be interested by flows defined on the torus $\mathbb{T}^d$.

Our concern is the resolution of the Cauchy problem for (2.1). Most PDEs require a form of boundary condition to be well-posed, and the Euler system is no different. But when working on the whole space $\mathbb{R}^d$, which has no boundary, one must rather require conditions at infinity in the form of a far field condition on the velocity: for example, we might demand the fluid to be at rest at infinity in some inertial reference frame given by a fixed vector $V \in \mathbb{R}^d$,

$$u(t, x) \longrightarrow V \qquad \text{as } |x| \to +\infty.$$

The absolute necessity of such a condition is made obvious by a simple example: consider $f \in C^\infty(\mathbb{R})$ as well as a fixed $e \in \mathbb{R}^d$ and define a flow by

$$u(t, x) = f(t) \qquad \text{and} \qquad \pi(t, x) = -f'(t) \cdot x. \tag{2.2}$$

[1] Simone Weil, *Note sur la suppression générale des partis politiques.*

Then the couple (u, π) is a solution of (2.1). But taking f to be any function that is compactly supported away from $t = 0$, we construct an infinity of solutions that correspond to the initial data $u(0) = 0$. And this lack of uniqueness cannot be dismissed as an artifact of the Galilean invariance of the equations: there is no inertial reference frame in which the fluid (2.2) is always at rest at infinity.

Note that this issue is of a different nature than the C^m or $C^1 \cap L^2$ ill-posedness displayed in [14] and [40]. For instance as the authors of [40] rely on singular integral operators not mapping $C^0 \cap L^2$ to L^∞, while our example (2.2) hinges on a lack of far field-conditions.

The questions we ask in this chapter are the following: *What type of far-field condition should a (smooth) solution of* (2.1) *satisfy to be uniquely determined by the initial data? Can such a condition be "optimal", in the sense that any other condition has to be stronger?*

Before diving into the rest of this chapter, let us note that bounded solutions of infinite energy, such as (2.2), are of specific interest in fluid mechanics: in particular, the space $W^{1,\infty}$ of Lipschitz functions is the largest space in which one can hope to solve most equations of ideal fluid mechanics –the first equation in (2.1) is nearly a transport equation. So far, all well-posedness results have been proved for solutions in spaces strictly embedded in $W^{1,\infty}$.

In addition, infinite energy solutions have also been at the center of much attention in the past twenty years, and present their own interesting challenges (see [56] for an introduction to these questions), so that a precise theory of non-integrable solutions seems quite desirable.

2.1.1 Role of the Pressure and Leray Projection

The analysis of this problem hinges on our understanding of the pressure force $-\nabla\pi$. An intuitive reason π has this key role can be seen from our counter-example. In (2.2), the flow is subject to a forcing: a pressure differential coming from $|x| \to +\infty$ induces the motion of the fluid. And of course, any fluid that is under the influence of an arbitrary forcing cannot have its dynamics solely determined by initial data.

Now, although the pressure is formally one of the unknowns of the system, it is entirely determined (up to a constant) by the velocity: ultimately, only the velocity field really matters when solving the system. In fact, we may formally compute $\nabla\pi$ as a function of u by taking the divergence of the first equation in (2.1) and solving the elliptic equation thus produced: we obtain

$$\nabla\pi = \nabla(-\Delta)^{-1} \sum_{j,k} \partial_j \partial_k (u_j u_k). \tag{2.3}$$

From the previous equation, we introduce the Leray projection operator

$$\mathbb{P} = \mathrm{Id} + \nabla(-\Delta)^{-1} \operatorname{div}$$

which may be seen as the L^2-orthogonal projector on the subspace of divergence-free functions. In other words, we may see the pressure term in $\nabla\pi$ in (2.1) as an orthogonal projection enforcing the divergence-free condition $\operatorname{div}(u) = 0$ at all times. Applying $\mathbb{P}$ to (2.1), we get a new equation

$$\partial_t u + \mathbb{P} \operatorname{div}(u \otimes u) = 0, \tag{2.4}$$

which is (up to a commutator term) a transport equation.

The importance of this new equation lies in that, unlike the original Euler system (2.1), the projected equation (2.4) is *well-posed* in a space of (regular enough) bounded functions: Pak and Park prove in 2004 [87] that for any Besov-Lipschitz initial datum $u_0 \in B^1_{\infty,1}$, there is a unique (local) solution $u \in C^0_T(B^1_{\infty,1})$ for some $T > 0$.

However, we see that systems (2.1) and (2.4) are not equivalent. For instance, the uniform flow $u(t, x) = f(t)$ of (2.2) is a solution of (2.1) but not of (2.4). While this may seem a bit surprising, general unpleasantness must be expected when dealing simultaneously with constant functions, as in (2.2), and Leray projection: constant functions are gradients and divergence-free functions at the same time.

The difference between (2.1) and (2.4) is that the projected equations contain an implicit far-field condition: the pressure force can only be expressed by Leray projection (2.3) under a (to be determined) far-field condition. The obvious reason for this is the presence of the inverted Laplace operator $(-\Delta)^{-1}$, which has its range in a space of functions with no harmonic component. Because the pressure is given by a Poisson equation[2]

$$-\Delta\pi = \partial_j\partial_k(u_j u_k),$$

it is determined in $\mathcal{S}'$ up to the addition of a harmonic polynomial. For every t, there is a $Q(t) \in \mathbb{R}[X]$ such that

$$\partial_t u + \mathbb{P}\,\mathrm{div}(u \otimes u) + \nabla Q = 0,$$

so that a solution u of the original Euler system (2.1) solves the projected system (2.4) if and only if $\nabla Q = 0$.

Regarding our question of whether a solution u of (2.1) may be determined by the initial datum u_0 alone, we see it depends on *knowing exactly when a solution of the Euler equations is a solution of the projected system* (2.4).

While this question, and the presence of the harmonic polynomial Q, is sometimes dismissed as being irrelevant, equation (2.3) giving the "canonical" choice of the pressure corresponding to a loosely stated far field condition, the question we ask (and solve) is much deeper, as we wish to know what property of the flow this choice is related to.

Remark 2.1. It should be noted that our problem is due to the fact that the solutions (u, π) are defined on the *non-compact* space $\mathbb{R}^d$. For the Euler problem set on the torus $\mathbb{T}^d$, the issue completely disappears. The pressure solves a Poisson equation, and hence is given by its Fourier coefficients

$$\forall k \in \mathbb{Z}^d \backslash \{0\}, \qquad \widehat{\pi}(k) = -\sum_{j,l} \frac{k_j k_l}{|k|^2} \widehat{u_j u_l}(k)$$

and a constant function (for $k = 0$), which is irrelevant as it does not change the value of the pressure force $-\nabla\pi$ (in other words, the mean value of the solution is preserved). Therefore, the velocity field does indeed solve the projected equation (2.4).

We point out that there is a slight difference between solutions defined on $\mathbb{T}^d$ and periodic flows on $\mathbb{R}^d$, as these last solutions, such as (2.2), may be driven by an exterior pressure. In that case, the pressure force $-\nabla\pi$ is periodic, although the pressure itself is not.

2.1.2 Previous Results

This question has been abundantly studied in the past twenty years, but mainly for the Navier-Stokes equations, although all existing results can be applied to the Euler system with suitable adaptations. We mention some of the notable advances on the topic.

The first kind of result is a condition on the pressure π for it to be given by (2.3). For instance, if π is of the form

$$\pi = \pi_0 + \sum_{i,j} R_i R_j \pi_{ij},$$

where the $R_k = \partial_k(-\Delta)^{-1/2}$ are the Riesz transforms and $\pi_{ij}(t), \pi_0(t) \in L^\infty$, then (2.3) holds, as proven in 2000 by Giga, Inui, J. Kato and Matsui [59]. Later, J. Kato [67] extended this result:

[2]From now on, we agree that there is an explicit summation on repeated indices. Here, for instance, there is a sum on $j, k = 1, ..., d$.

in order for (2.3) to be true, one only needs $\pi(t) \in \mathrm{BMO}$. Other results include those of Kukavica and Vicol [72], who require that $|\pi(t,x)| = o(|x|)$ as $|x| \to +\infty$, or Maremonti [82] with a finite moment condition $(1+|x|)^{d+1}\pi(t,x) \in L^1$. Finally, Nakai and Yoneda [85] introduce a condition based on Campanato-type spaces.

A second type of theorem shows that any solution u of the Euler system (2.1) must be given by transformation of a solution v of (2.4) through a "generalized Galilean transform":

$$u(t,x) = v\big(t, x - G(t)\big) + g(t) \qquad \text{where } G(t) = \int_0^t g(s)\mathrm{d}s. \tag{2.5}$$

See for example the articles of Kukavica [71] and Kukavica and Vicol [72]. The generalized Galilean transform places the fluid in an accelerated reference frame in which an inertial force appears. This force can be seen as a pressure differential $\nabla(g'(t)\cdot x)$, and interpreted as a forcing term.

Finally, a last type of result refers to the velocity field only: a far-field condition based on $u(t,x)$ is sufficient to constrain the form of the pressure. For instance, in [29], we have proven that a loose integrability condition at $|x| \to +\infty$ is enough for (2.3) to be true. Similarly, the recent work of Fernández-Dalgo and Lemarié-Rieusset [54], which explores the problem with much detail, provides another sufficient condition based on a finite momentum condition: the flow is required to satisfy

$$\forall T > 0, \qquad \int_0^T \int \frac{|u(t,x)|^2}{(1+|x|)^d}\,\mathrm{d}x\,\mathrm{d}t < +\infty. \tag{2.6}$$

Finally, we point out a last result, which is given by Lemarié-Rieusset in [77] (point *ii* of Theorem 11.1, pp. 109-111), where the flow is required to satisfy the Morrey-type condition

$$\forall t_1 < t_2, \qquad \sup_{x\in\mathbb{R}^d} \frac{1}{\lambda^d}\int_{t_1}^{t_2}\int_{|x-y|\le\lambda} \big|u(t,y)\big|^2\mathrm{d}y\,\mathrm{d}t \underset{\lambda\to+\infty}{\longrightarrow} 0. \tag{2.7}$$

These last conditions of [29], [54] and [77], although they are very general, cannot be optimal (both sufficient and neccessary). There are solutions of the projected problem (2.4) which do not fulfill conditions (2.6), (2.7) or the one in [29]. For instance, these conditions fail for any smooth nonzero periodic solution of (2.4), such as *e.g.* the constant flow $u(t,x) = Cst$.

Finally, we refer to [68] for various results concerning the uniqueness of Serfati flows (see [97], [3]) as solutions of the Euler equations. In Subsection 2.4.2, we discuss some of these and their relation with our own result.

2.1.3 Main result

We now present and discuss the main theorem of this chapter. In order to introduce the formal ideas behind our main result, Theorem 2.2 below, consider a smooth bounded solution u of the Euler system which has bounded derivatives. As we have explained above, the pressure field is a solution of a Poisson equation

$$-\Delta\pi = \partial_j\partial_k(u_ju_k)$$

and the issue lies in the fact that the solutions of this elliptic problem are not unique, as they are given, in $\mathcal{S}'$, up to the addition of a harmonic polynomial. Therefore, in order to recover the pressure force $-\nabla\pi$, we must add to (2.3) the gradient of a harmonic polynomial $Q \in \mathbb{R}[X]$, so as to obtain

$$\partial_t u + \mathbb{P}\,\mathrm{div}(u\otimes u) + \nabla Q = 0. \tag{2.8}$$

Therefore, u will be a solution of the projected system (2.4) if and only if $\nabla Q \equiv 0$.

A crucial element of our argument is that polynomial functions are spectrally supported at $\xi = 0$. This means that to determine whether ∇Q is nonzero or not, it is enough to study the *low frequency* behavior of (2.8). Consider a nonnegative and compacty supported cut-off function χ with $\chi(\xi) = 1$ around $\xi = 0$. Then, by applying $\chi(\lambda D)$ to (2.8) for $\lambda > 0$, we get

$$\partial_t \chi(\lambda D) u + \chi(\lambda D) \mathbb{P} \operatorname{div}(u \otimes u) + \nabla Q = 0,$$

and by letting $\lambda \to +\infty$, we hope to recover information on ∇Q based on the properties of u. Focus for the moment on the convective term $\chi(\lambda D)\mathbb{P}\operatorname{div}(u \otimes u)$. The symbol of the operator $\mathbb{P}\operatorname{div}$ is a homogeneous function of order 1, and so is $O(|\xi|)$ at low frequencies. It is therefore extremely tempting to resort to the first Bernstein inequality (Lemma 1.7) and write that the low frequency limit reads

$$\left\| \chi(\lambda D) \mathbb{P} \operatorname{div}(u \otimes u) \right\|_{L^\infty} = O\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda \to +\infty. \tag{2.9}$$

Unfortunately, the Bernstein inequalities do not apply to the symbol of $\mathbb{P}\operatorname{div}$ because it lacks regularity: its derivatives are not continuous (see Remark 1.8). Nevertheless, we will see that it is still possible to show that (2.9) is true, but by using more involved computations. The convergence (2.9), which implies that $\partial_t \chi(\lambda D) u \longrightarrow \nabla Q$, shows that $\nabla Q \equiv 0$ if and only if

$$\chi(\lambda D) \partial_t u \longrightarrow 0 \qquad \text{as } \lambda \to +\infty$$

at all times, or in other words $\partial_t u \in \mathcal{S}'_h$ (see Definition 1.3). We will prove the following equivalence theorem, whose statement is a natural outcome of the discussion above.

Theorem 2.2. *Let $T > 0$, $u_0 \in L^\infty$ and $u \in C^0([0, T[; L^\infty)$ be a weak solution of the Euler equations* (2.1) *associated to the initial datum u_0 and to a pressure $\pi \in \mathcal{D}'([0, T[\times \mathbb{R}^d)$. Then the following assertions are equivalent:*

(i) the flow u solves the projected problem with initial datum $u(0)$ that satisfies $u(0) - u_0 \in \mathrm{Cst} \in \mathbb{R}$,

(ii) for all times $t \in [0, T[$, we have $u(t) - u(0) \in \mathcal{S}'_h$,

(iii) for all times $t \in [0, T[$, we have $u(t) - u(0) \in \mathrm{BMO}^{-1}$,

(iv) the pressure satisfies $\pi \in C^0(]0, T[; \mathrm{BMO})$,

(v) the pressure force is continuous with respect to time $\nabla \pi \in C^0(]0, T[; \mathcal{S}')$ and $\nabla \pi(t) \in \mathcal{S}'_h$ for all $0 < t < T$.

Remark 2.3. The distinction made in assertion *(i)* between the initial datum u_0 and the initial value $u(0)$ may seem unusual. The initial value problem for bounded solutions has a few difficulties: initial data are only defined up to an additive constant. However, these technicalities are not at the core of the proof.

Remark 2.4. In assertion *(ii)*, the condition $u(t) - u(0) \in \mathcal{S}'_h$ is simply an integrated version of the one $\partial_t u \in \mathcal{S}'_h$ we used above.

Remark 2.5. A small clarification must be made concerning point *(iv)* in Theorem 2.2. Because $\mathrm{BMO} \subset L^1_{\mathrm{loc}} / \mathbb{R}$ is a space of functions defined modulo constants, the condition $\pi \in C^0_T(\mathrm{BMO})$ does not mean that $\pi(t, x)$ is continuous with respect to time as a distribution, even in the $\mathcal{D}'$ topology.

Let us compare Theorem 2.2 with the other known results we mentioned. Firstly, unlike the ones we presented in Subsection 2.1.2, Theorem 2.2 is *optimal* in the sense that we provide a *necessary and sufficient* condition for a bounded solution of the original Euler equations (2.1) to solve the projected equations (2.4).

Next, amongst the various results discussed in Subsection 2.1.2, we see that the condition $\pi(t) \in \mathrm{BMO}$ of J. Kato [67] is optimal.[3] On the one hand, if $u \in L^\infty$ is a solution of the projected problem (2.4), then π is the image of a bounded function by a singular integral operator $\pi = (-\Delta)^{-1}\partial_j\partial_k(u_j u_k)$, and hence lies in BMO. On the other hand, if $\pi(t) \in \mathrm{BMO}$ then the pressure force $\nabla\pi$ is in $\mathrm{BMO}^{-1} \subset \mathcal{S}'_h$, so ∇Q must be zero.

The condition $\pi(t,x) = o(|x|)$ of Kukavica and Vicol [72] is also weaker than assertion *(v)*. This is a consequence of the following lemma.

Lemma 2.6. *Let $f \in L^\infty_{\mathrm{loc}}$ be a function such that $f(x) = o(|x|)$ as $|x| \to +\infty$. Then $\nabla f \in \mathcal{S}'_h$.*

Proof. First, write $f(x) = |x|\epsilon(x)$ with $\epsilon(x) \longrightarrow 0$ as $|x| \to +\infty$. In order to take advantage of this, we use the convolution product form of $\chi(\lambda D)\nabla f$. Fix $\psi_\lambda(x) = \lambda^{-d}\psi(\lambda^{-1}x)$ such that $\widehat{\psi_\lambda}(\xi) = \chi(\lambda\xi)$. We have

$$\chi(\lambda D)\nabla f(x) = \nabla\psi_\lambda * f(x) = \frac{1}{\lambda}\int |y|\epsilon(y)\nabla\psi\left(\frac{x-y}{\lambda}\right)\frac{\mathrm{d}y}{\lambda^d}.$$

By changing variables in this integral, we obtain the upper bound

$$|\nabla\psi_\lambda * f(t,x)| \leq \int |y|\epsilon(\lambda y)\left|\nabla\psi\left(\frac{x}{\lambda} - y\right)\right|\mathrm{d}y,$$

which tends to zero uniformly locally as $\lambda \to +\infty$ by dominated convergence. Therefore, we also have

$$\chi(\lambda D)\nabla f \underset{\lambda\to+\infty}{\longrightarrow} 0 \qquad \text{in } \mathcal{S}'.$$

□

Remark 2.7. In some sense, the condition $\pi(t,x) = o(|x|)$ also seems to be somewhat optimal: because of inequality (1.13), the pressure $\pi(t) \in \mathrm{BMO}$ should have roughly at most logarithmic growth. Also compare this idea to the growth of functions implied by the John-Nirenberg inequality.

Finally, we comment on the last conditions (2.6) and (2.7), which bear on the velocity field. It must be noted that the framework of [54] and [77] is, in a sense, more general because it deals with Navier-Stokes solutions that have locally finite energy, whereas only more regular Besov-Lipschitz solutions make sense for our ideal fluid equations. On the other hand, both (2.7) and (2.6) are particular cases of $\mathcal{S}'_h$ functions.

Lemma 2.8. *Consider $f \in L^\infty$. Assume that either one of the following conditions hold:*

$$\int \frac{|f(x)|^2}{(1+|x|)^d}\,\mathrm{d}x < +\infty \qquad \text{or} \qquad \frac{1}{\lambda^d}\int_{|x-y|\leq\lambda} |f(y)|^2\mathrm{d}y \underset{\lambda\to+\infty}{\longrightarrow} 0 \quad \text{in } L^\infty_{\mathrm{loc}}.$$

Then $f \in \mathcal{S}'_h$. In fact, the first condition implies the second one.

[3]Here and below, we must point out remember that the results of [67] and [72] are originally stated for the Navier-Stokes equations in different time regularities.

Proof. We start by showing that the first moment-type condition implies the second Morrey-type one. For all $\lambda > 0$, we have

$$\frac{1}{\lambda^d}\int_{|x-y|\leq\lambda} |f(y)|^2 \mathrm{d}y = \int_{|x-y|\leq\lambda} \left(\frac{1+|y|}{\lambda}\right)^d \frac{|f(y)|^2}{(1+|y|)^d}\,\mathrm{d}y.$$

Set $g(y) = (1+|y|)^{-d}|f(y)|^2$ and consider $r > 0$ and $\epsilon > 0$. Because $g \in L^1$, we may fix a $R > 0$ such that

$$\forall |x| \leq r, \qquad \int_{|x-y|\geq R} g(y)\,\mathrm{d}y \leq \epsilon.$$

Therefore, by separating in our integral the $y \in \mathbb{R}^d$ that are close to x from those that are far away, we have, for all $|x| \leq r$,

$$\begin{aligned}
\frac{1}{\lambda^d}\int_{|x-y|\leq\lambda} |f(y)|^2 \mathrm{d}y &= \int_{|x-y|\leq R} \left(\frac{1+|y|}{\lambda}\right)^d g(y)\,\mathrm{d}y \ + \int_{R\leq|x-y|\leq\lambda} \left(\frac{1+|y|}{\lambda}\right)^d g(y)\,\mathrm{d}y \\
&\leq \int_{|x-y|\leq R} \left(\frac{1+|y|}{\lambda}\right)^d g(y)\,\mathrm{d}y \ + \epsilon \\
&\leq \left(\frac{1+|x|+R}{\lambda}\right)^d \|g\|_{L^1} + \epsilon.
\end{aligned}$$

The quantity in the last line converges uniformly locally to zero, so we have the desired L^∞_{loc} convergence. Finally, we only have to show that the second condition implies $f \in \mathcal{S}'_h$. Consider $\psi_\lambda \in \mathcal{S}$ as in the proof of Lemma 2.6 above and let $\epsilon > 0$. We fix a $R > 0$ such that $\|\mathbb{1}_{|y|\geq R}\psi(y)\|_{L^1} \leq \epsilon$. We have

$$|\chi(\lambda D)f(x)| \leq \int_{|x-y|\leq\lambda R} |\psi_\lambda(x-y)f(y)|\,\mathrm{d}y \ + \int_{|x-y|\geq\lambda R} |\psi_\lambda(x-y)f(y)|\,\mathrm{d}y.$$

Because $\psi \in \mathcal{S}$ is a Schwartz function and $f \in L^\infty$, the second integral is bounded by

$$\begin{aligned}
\int_{|x-y|\geq\lambda R} |\psi_\lambda(x-y)f(y)|\,\mathrm{d}y &\leq \|f\|_{L^\infty}\int_{|x-y|\geq\lambda R} |\psi_\lambda(x-y)|\,\mathrm{d}y \\
&= \|f\|_{L^\infty}\|\mathbb{1}_{|y|\geq R}\psi(y)\|_{L^1} \leq \epsilon\|f\|_{L^\infty}.
\end{aligned}$$

On the other hand, we may apply the Cauchy-Schwarz inequality to the first integral, so as to obtain

$$\begin{aligned}
\int_{|x-y|\leq\lambda R} |\psi_\lambda(x-y)f(y)|\,\mathrm{d}y &\leq \left(\int_{|x-y|\leq\lambda R} |f(y)|^2\,\mathrm{d}y\right)^{1/2} \left(\int_{|x-y|\leq\lambda R} |\psi_\lambda(x-y)|^2\,\mathrm{d}y\right)^{1/2} \\
&\leq \left(\frac{1}{\lambda^d}\int_{|x-y|\leq\lambda R} |f(y)|^2\,\mathrm{d}y\right)^{1/2} \left(\int \left|\psi\left(\frac{x-y}{\lambda}\right)\right|^2 \frac{\mathrm{d}y}{\lambda^d}\right)^{1/2} \\
&= \|\psi\|_{L^2}\left(\frac{1}{\lambda^d}\int_{|x-y|\leq\lambda R} |f(y)|^2\,\mathrm{d}y\right)^{1/2}.
\end{aligned}$$

Since, by assumption, this quantity converges uniformly locally to zero, it also does so in the $\mathcal{S}'$ topology. □

Remark 2.9. In fact, the results of Lemma 2.8 hold when f is in the Morrey space of uniformly locally L^2 functions (see Definition 11.4 p. 108 in [77]) with pretty much the same proofs. These are the $f \in L^2_{\mathrm{loc}}$ such that the integrals

$$\int_B |f(x)|^2\,\mathrm{d}x \leq C$$

are bounded for all balls $B \subset \mathbb{R}^d$ by a constant that depends only on the volume of the balls $C = C(|B|)$. But we will not need such a level of generality.

Remark 2.10. There are many functions in $\mathcal{S}'_h$ which do not fulfill the conditions of Lemma 2.8. For instance, any periodic function with average value zero is in $\mathcal{S}'_h$ without any of the properties of Lemma 2.8 being true. Similarly, we have shown in Chapter 1, Example 1.34, that the sign function σ is $\mathcal{S}'_h$, while it is obvious that none of the properties of the Lemma are fulfilled.

2.2 Bounded Weak Solutions of the Euler System

In this paragraph, we clarify the precise meaning of a bounded weak solution of the Euler system (2.1) and the projected problem (2.4). This will require exploring further the properties of the operator $\mathbb{P}\,\mathrm{div}$.

2.2.1 Weak Solutions of the Euler Equations

We define and comment the notion of bounded weak solutions of the Euler equations: these read

$$\begin{cases} \partial_t u + \mathrm{div}(u \otimes u) + \nabla \pi = 0 \\ \mathrm{div}(u) = 0. \end{cases} \tag{2.10}$$

Definition 2.11. Let $u_0 \in L^\infty$ be a divergence-free function and $T > 0$. A function $u \in L^2_{\mathrm{loc}}([0,T[;L^\infty)$ is said to be a *bounded weak solution* of the Euler problem (2.10) associated to the initial datum u_0 if there is a pressure $\pi \in \mathcal{D}'([0,T[\times\mathbb{R}^d)$ such that

$$\begin{cases} \partial_t u + \mathrm{div}(u \otimes u) + \nabla \pi = \delta_0(t) \otimes u_0(x) \\ \mathrm{div}(u) = 0 \end{cases} \qquad \text{in } \mathcal{D}'([0,T[\times\mathbb{R}^d). \tag{2.11}$$

In the equation above, the tensor product $\delta_0(t) \otimes u_0(x)$ is defined by the relation

$$\forall \phi \in \mathcal{D}([0,T[\times\mathbb{R}^d), \qquad \big\langle \delta_0(t) \otimes u_0(x), \phi(t,x) \big\rangle := \int u_0(x) \cdot \phi(0,x)\,\mathrm{d}x.$$

Remark 2.12. Several remarks are in order concerning this definition. Firstly, by testing (2.11) against any divergence-free $\phi \in \mathcal{D}([0,T[\times\mathbb{R}^d)$, we obtain the usual integral formulation

$$\int_0^T \int \Big\{ \partial_t \phi \cdot u + \nabla \phi : u \otimes u \Big\}\,\mathrm{d}x\,\mathrm{d}t + \int u_0 \cdot \phi(0)\,\mathrm{d}x = 0 \tag{2.12}$$

of the momentum equation. Strictly speaking, Definition 2.11 is a bit stronger than the previous integral formulation because an extra assumption is made on the pressure: it has to define a distribution with respect to both *time and space*.

Remark 2.13. Next, we note that, contrary to most evolution equations, there is no need for a weak solution u to be continuous with respect to time, even in the $\mathcal{D}'$ topology: the presence of the pressure may compensate time singularities in the flow, as would be the case if, for example, $u(t,x) = \mathbb{1}_{[1,+\infty[}(t)V$ where $V \in \mathbb{R}^d$ is a constant vector. As we will see, this problem arises only in the L^∞ framework: L^p regularity of the solutions (with $p < +\infty$) implies continuity with respect to time in some weak topology, since, in that case, we may indeed apply the Leray projection to obtain (2.13) below.

Remark 2.14. Finally, we must comment on the notion of initial datum in Definition 2.11. Because the weak form of the Euler equations is taken in the sense of $\mathcal{D}'([0,T[\times\mathbb{R}^d)$, that is with

$t = 0$ included in the time interval, any translation $u_0 + V$ (with $V \in \mathbb{R}^d$) of the initial datum is also a valid initial datum *for the same solution*:

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla\big(\pi + \delta_0(t) \otimes V\big) = \delta_0(t) \otimes \big(u_0(x) + V\big).$$

Simply, a different choice of pressure allows for u to be associated with infinitely many initial data, regardless of the time regularity of u (it could be $C^1_T(L^\infty)$). Seen from the perspective of the integral form (2.12) of the momentum equation, the ambiguity of the initial data is a direct consequence of testing with *divergence-free* functions only: for any fixed $V \in \mathbb{R}^d$ and divergence-free $\phi \in \mathcal{D}([0, T[\times\mathbb{R}^d)$,

$$\int (u_0 + V) \cdot \phi(0)\,\mathrm{d}x = \int u_0 \cdot \phi(0)\,\mathrm{d}x.$$

These test functions are orthogonal, in the sense of the $\mathcal{S}' \times \mathcal{S}$ duality, to all gradients. Therefore, any (regular) flow u which solves the Euler problem is also a weak solution (according to Definition 2.11) related to the initial data $u(0) + \nabla g$ (for any $g \in \mathcal{S}'$).

In the L^p framework (with $p < +\infty$), this ambiguity totally disappears, as long as we require the initial data to be L^p. Indeed, any divergence-free $f \in L^p$ which is also the gradient of some tempered distribution $f = \nabla g$ must be a harmonic polynomial, since we would then have $\Delta g = 0$. Since there is no nonzero polynomial in L^p, this implies $f = 0$.

Our problem lies in the fact that the space L^∞ intersects non-trivially with $\mathbb{R}[X]$, so that there are nonzero divergence-free functions which are also gradients: constant functions.

As we said in the introduction, bounded solutions of the Euler equations (2.10) are not unique, even under C^∞ smoothness assumptions.

2.2.2 Weak Solutions of the Projected Problem

In this section, we focus on the projected problem: formally, one may apply the Leray projection operator to kill the pressure term and obtain an equation on u only, namely

$$\partial_t u + \mathbb{P}\operatorname{div}(u \otimes u) = 0. \tag{2.13}$$

However, as we saw in Chapter 1, Fourier multiplication operators with bounded homogeneous symbols of degree zero, such as the Leray projection $\mathbb{P}$, are not properly defined on L^∞. We must therefore clarify the meaning of equation (2.13) when the solutions are solely bounded, say $u \in L^2_T(L^\infty)$.

The general idea of this paragraph is to study the order one operator $\mathbb{P}\operatorname{div}$ instead of $\mathbb{P}$, which is a combination of operators the form $T_{j,k,l} := \partial_j\partial_k\partial_l(-\Delta)^{-1}$. By introducing the fundamental solution $E \in L^1_{\mathrm{loc}}$ of the Laplacian on $\mathbb{R}^d$ given by

$$E(x) = \frac{C(d)}{|x|^{d-2}} \text{ if } d \geq 3 \qquad \text{and} \qquad E(x) = C(2)\log|x| \text{ if } d = 2,$$

where the constant $C(d) > 0$ is such that $-\Delta E = \delta_0$, we may see $T_{j,k,l}$ as a convolution operator whose kernel is $\Gamma_{j,k,l} := \partial_j\partial_k\partial_l E$ (seen as a distribution). We prove the following Proposition, which is a decomposition of Lemarié-Rieusset [77] (see Lemma 11.1 p. 106) also used in the simpler setting of bounded solutions by Pak and Park [87] (see equations (20) and (21) p. 1161).

Proposition 2.15. *Consider $j, k, l \in \{1, ..., d\}$ and let $\Gamma_{j,k,l} = \partial_j\partial_k\partial_l E \in \mathcal{D}'$. Then $\Gamma_{j,k,l}$ is the sum of a compactly supported distribution and an L^1 function. In addition, for all $s \in \mathbb{R}$ and $p, r \in [1, +\infty]$, convolution by Γ defines a bounded operator on the non-homogeneous Besov spaces*

$$T_{j,k,l} : B^s_{p,r} \longrightarrow B^{s-1}_{p,r}.$$

Remark 2.16. The Besov boundedness of $T_{j,k,l} : B^s_{p,r} \longrightarrow B^{s-1}_{p,r}$ is falsely obvious in the case $p = +\infty$. Although Lemma 1.5 makes it clear that the operator

$$\tilde{T}_{j,k,l} := \sum_{m \in \mathbb{Z}} \dot{\Delta}_m \partial_j \partial_k \partial_l (-\Delta)^{-1}$$

is bounded in the $B^s_{\infty,r} \longrightarrow B^{s-1}_{\infty,r}$ topology, as the sum converges normally in L^∞ when restricted to ranks $j \leq -1$, the two operators $\tilde{T}_{j,k,l}$ and $T_{j,k,l}$ might not coincide: they could differ by a polynomial. This is why we need two extra steps: firstly to define $T_{j,k,l}$, this is the purpose of Proposition 2.15, and secondly to show that $\tilde{T}_{j,k,l}$ and $T_{j,k,l}$ are actually equal, this being the scope of Proposition 2.24 below.

Proof. By fixing a compactly supported function $\chi \in \mathcal{D}$ with $\chi(x) \equiv 1$ around $x = 0$, we write $\Gamma_{j,k,l} = \chi \partial_j \partial_k \partial_l E + (1 - \chi) \partial_j \partial_k \partial_l E$. Because of the form of the fundamental solution E, its third derivatives are integrable at infinity:

$$\partial_j \partial_k \partial_l E(x) = O\left(\frac{1}{|x|^{d+1}}\right) \qquad \text{as } |x| \to +\infty. \tag{2.14}$$

Therefore, the function $(1 - \chi)\partial_j \partial_k \partial_l E$ is L^1 and we have proved that $\Gamma_{j,k,l} \in \mathcal{E}' + L^1$, where $\mathcal{E}'$ is the space of compactly supported distributions. Concerning boundedness in Besov spaces, we note that the operator $\mathbb{P}\,\mathrm{div}$ is entirely defined for frequencies $\xi \neq 0$ by the Fourier transform

$$\forall \phi \in \mathcal{D}(\mathbb{R}^d \setminus \{0\}; \mathbb{R}^d \otimes \mathbb{R}^d), \qquad \mathcal{F}\big[\mathbb{P}\,\mathrm{div}\,\mathcal{F}^{-1}[\phi]\big](\xi) = \sigma(\xi)\phi(\xi),$$

where σ is the symbol of $\mathbb{P}\,\mathrm{div}$. Consequently, Lemma 1.5 makes it clear that for any $m \geq 0$ and $f \in B^s_{p,r}$ we have

$$\|\Delta_m \mathbb{P}\,\mathrm{div}(f)\|_{L^p} \leq C 2^m \|\Delta_m f\|_{L^p}.$$

It only remains to estimate the low frequency block $\Delta_{-1}\mathbb{P}\,\mathrm{div}(f)$. Let $\psi \in \mathcal{S}$ such that the operator Δ_{-1} is the convolution by ψ, that is $\Delta_{-1} f = \psi * f$. We use the cut-off function θ as above and split the kernel into two parts

$$\Delta_{-1}\mathbb{P}\,\mathrm{div}(f) = \psi * \Gamma_{j,k,l} * f = \psi * \partial_j \partial_k \partial_l (\theta E) + \psi * \partial_j \partial_k \partial_l \big((1 - \theta)E\big).$$

Integrating by parts in the convolution product yields

$$\psi * \Gamma_{j,k,l} * f = \partial_j \partial_k \partial_l \psi * (\theta E) * f + \psi * \partial_j \partial_k \partial_l \big((1 - \theta)E\big) * f.$$

Now, because E is locally integrable and thanks to (2.14), both kernels

$$\partial_j \partial_k \partial_l \psi * (\theta E) \in L^1 \qquad \text{and} \qquad \psi * \partial_j \partial_k \partial_l \big((1 - \theta)E\big) \in L^1$$

are integrable functions. The Hausdorff convolution inequality guarantees that $\Delta_{-1}\mathbb{P}\,\mathrm{div}$ is a bounded $L^p \longrightarrow L^p$ operator for all $p \in [1, +\infty]$. $\square$

Proposition 2.15 makes it possible to define the notion of bounded weak solution of the projected Euler system (2.13).

Definition 2.17. Let $T > 0$ and $u_0 \in L^\infty$ be a bounded divergence free function. We say that $u \in L^2_{\mathrm{loc}}([0, T[; L^\infty)$ is a *bounded weak solution* of the projected Euler system (2.13) associated to the initial datum u_0 if

$$\partial_t u + \mathbb{P}\,\mathrm{div}(u \otimes u) = \delta_0(t) \otimes u_0(x) \qquad \text{in } \mathcal{D}'([0, T[\times \mathbb{R}^d).$$

Bounded weak solutions of the projected system (2.13) behave in many ways better than weak solutions of the Euler system. For example, solutions are necessarily continuous with respect to time in the $B^{-1}_{\infty,\infty}$ topology. In addition, the pressure, which is given by

$$\pi = (-\Delta)^{-1}\partial_j\partial_k(u_j u_k) \qquad \text{(modulo constants)} \tag{2.15}$$

has minimal regularity properties: being the image of the bounded function $u_j u_k \in C^0_T(L^\infty)$ by the singular integral operator $(-\Delta)^{-1}\partial_j\partial_k$, it defines a BMO function

$$\pi \in C^0([0,T[;\mathrm{BMO}).$$

However, it must be noted that because BMO is a space of functions defined up to a constant (and hence $\mathrm{BMO} \hookrightarrow \mathcal{S}'/\mathbb{R}[X]$), the pressure does not define a distribution on $[0,T[\times\mathbb{R}^d$, as the constant summand in (2.15) may be extremely singular in the time variable. On the other hand, because the pressure force is the derivative of (2.15), it unambiguously defines a $\mathrm{BMO}^{-1} \subset \mathcal{S}'$ function

$$\nabla\pi \in C^0([0,T[;\mathrm{BMO}^{-1}) \subset \mathcal{D}'([0,T[\times\mathbb{R}^d).$$

2.3 Proof the Main Result

We are now ready to prove our main result, Theorem 2.2. We will start by recalling the statement of Theorem 2.2 and making a few additional comments. Then, the proof will start with the simpler case where the solutions are C^1 with respect to the time variable, before dealing with the full statement of Theorem 2.2.

2.3.1 Statement of the Theorem and Comments

For the readers convenience, we recall the statement of our main result.

Theorem 2.18. *Let $T > 0$, $u_0 \in L^\infty$ and $u \in C^0([0,T[;L^\infty)$ be a weak solution of the Euler equations* (2.10) *associated to the initial datum u_0 and to a pressure $\pi \in \mathcal{D}'([0,T[\times\mathbb{R}^d)$. Then the following assertions are equivalent:*

- *(i) the flow u solves the projected problem with initial datum $u(0)$ that satisfies $u(0) - u_0 \in \mathrm{Cst}$,*
- *(ii) for all times $t \in [0,T[$, we have $u(t) - u(0) \in \mathcal{S}'_h$,*
- *(iii) for all times $t \in [0,T[$, we have $u(t) - u(0) \in \mathrm{BMO}^{-1}$,*
- *(iv) the pressure satisfies $\pi \in C^0(]0,T[;\mathrm{BMO})$,*
- *(v) the pressure force is continuous with respect to time $\nabla\pi \in C^0(]0,T[;\mathcal{S}')$ and $\nabla\pi(t) \in \mathcal{S}'_h$ for all $0 < t < T$.*

Remark 2.19. In this Theorem, the velocity field is assumed to be continuous with respect to time in the L^∞ topology. This may seem odd when compared to known solutions of the Euler equations. For example, the solutions of Pak and Park [87] are continuous in $B^1_{\infty,1}$, and therefore, by Proposition 2.15,

$$\partial_t u = -\mathbb{P}\,\mathrm{div}(u \otimes u) \in C^0(B^0_{\infty,1}), \tag{2.16}$$

so that a $C^1(L^\infty)$ assumption might seem more natural, in addition to (as we will see) simplifying the proof. Unfortunately, exchanging space regularity for time regularity as we just did, requires having a solution of the projected problem. If $u \in C^0_T(B^1_{\infty,1})$ is only a solution of the Euler equations (2.10), one cannot repeat operation (2.16), as the pressure force $-\nabla\pi$ may well not be regular with respect to time. Recall that the uniform flow (2.2) solves the Euler problem even if $f(t)$ is not C^1.

In that regard, working with $C^1_T(L^\infty)$ solutions would be somewhat artificial. In contrast, continuity in the time variable is a very reasonable assumption, in the light of the existence of paradoxical solutions (that dissipate kinetic energy) in the class $C^0(L^2)$, see [35]. In other words, there is no point in going below time-continuous regularity.

Remark 2.20. We note that condition *(ii)* of the theorem is in fact a Galilean condition: its validity is independent of the inertial reference frame in which the velocity is computed. This shows that the equivalence issue has something to do with a fundamental property of the solution, and is not a mere artifact of the Galilean nature of the equations.

2.3.2 Equivalence of the Two Formulations: Smooth in Time Solutions

In this subsection, we prove Theorem 2.18 under a comfortable assumption that the solutions are $C^1(L^\infty)$. This simplifies the proof very much, as it will allow the derivative $\partial_t u$ to be handled as a locally integrable function.

Proposition 2.21. *Consider $T > 0$. Let $u_0 \in L^\infty$ be a divergence-free initial datum and $u \in C^1([0,T[;L^\infty)$ be a weak solution of the Euler problem (according to Definition 2.11) for some pressure field $\pi \in \mathcal{D}'([0,T[\times\mathbb{R}^d)$ and related to the initial datum u_0. Then all conditions of Theorem 2.18 are equivalent.*

The proof of this proposition is split in several steps. We start by reformulating the problem in a way that lets the Leray projector appear in the equations. This is the purpose of the following lemma.

Lemma 2.22. *Let u be as in Proposition 2.21. For all times $t \in]0,T[$, there exists a polynomial $Q(t) \in \mathbb{R}[X]$ such that*

$$\partial_t u + \mathbb{P}\,\mathrm{div}(u\otimes u) + \nabla Q(t) = 0. \tag{2.17}$$

Moreover, $Q(t)$ must be at most one linear: $\deg\big(Q(t)\big) \leq 1$.

Remark 2.23. A consequence of Lemma 2.22 is that any solution of the Euler system (2.10) is obtained from a solution of the projected system (2.13) from a “generalized Galilean transformation”, or in other words writing the projected equations (2.13) in an accelerated reference frame. It is mainly a matter of taking $g(t) = \nabla Q(t)$ in (2.5). We refer to [72] for precise arguments.

Proof of Lemma 2.22. Since $u \in C^1_T(L^\infty)$ is a solution of the Euler equations for the pressure π, we may write

$$\partial_t u + \mathrm{div}(u\otimes u) + \nabla\pi = 0 \qquad \text{in } \mathcal{D}'(]0,T[\times\mathbb{R}^d).$$

Now we fix once and for all a time $t \in]0,T[$. By taking the divergence of this equation, we see that the pressure solves the elliptic equation

$$-\Delta\pi = \partial_j\partial_k(u_j u_k),$$

in which there is an implicit sum on the repeated indices. We wish to invert this equation in order to recover π. In order to avoid the singularity of the inverse Laplacian operator $(-\Delta)^{-1}$ at the frequency $\xi = 0$, we localize away from low frequencies. For any $\phi \in \mathcal{S}$ such that $\widehat{\phi} \in \mathcal{D}(\mathbb{R}^d\setminus\{0\})$, we have

$$\big\langle \nabla\pi, \phi\big\rangle_{\mathcal{S}'\times\mathcal{S}} = \big\langle \nabla(-\Delta)^{-1}\partial_j\partial_k(u_j u_k), \phi\big\rangle_{\mathcal{S}'\times\mathcal{S}},$$

where the quantity $\nabla(-\Delta)^{-1}\partial_j\partial_k(u_ju_k) = T_{j,k}(u_j, u_k)$ it to be understood in the sense of Proposition 2.15. Therefore, the difference between the two distributions $\nabla\pi$ and $T_{j,k}(u_ju_k)$ is spectrally supported at $\xi = 0$, and must be a polynomial: there exists a $\nabla Q(t) \in \mathbb{R}[X]$ such that

$$\nabla\pi = -(\mathrm{Id} - \mathbb{P})\operatorname{div}(u \otimes u) + \nabla Q,$$

and the polynomial ∇Q is indeed a gradient function because all other terms in this equation are gradients. We have shown that (2.17) holds. The assertion on the degree of $Q(t)$ is obtained by applying the low frequency block Δ_{-1} to (2.17) and noting that

$$-\nabla Q = \partial_t\Delta_{-1}u + \Delta_{-1}\mathbb{P}\operatorname{div}(u \otimes u) \in C^0_T(L^\infty)$$

is at all times bounded in view Proposition 2.15, thus ending the proof of the lemma. □

Before moving onwards, we study the Leray projection more closely. The following proposition shows that, for any $f \in L^\infty$, the function $(\mathrm{Id} - \mathbb{P})\operatorname{div}(f)$ is in $\mathcal{S}'_h$.

Proposition 2.24. *Let Γ be as defined in Proposition 2.15. Then we have, for all $p \in [1, +\infty]$ and all $f \in L^p$, the estimate*

$$\big\|\chi(\lambda D)\big(\Gamma * f\big)\big\|_{L^p} = O\left(\frac{1}{\lambda}\log(\lambda)\right) \qquad \text{as } \lambda \to +\infty.$$

The factor $\log(\lambda)$ *may be dispensed with whenever* $d \geq 3$.

Remark 2.25. As mentioned above, the $O\left(\lambda^{-1}\right)$ estimate (neglecting the logarithmic term when $d = 2$) seems rather natural, considering that convolution by Γ is formally the Fourier multiplication by a homogeneous symbol of order 1. However, the first Bernstein inequality is insufficient to obtain this result, as, though it may be generalized to Fourier multipliers, it requires their symbol to be smooth.

Remark 2.26. In fact, Proposition 2.24 could be replaced by a much shorter but much more advanced argument: assume that $f \in L^p$ with $p > 1$. If $1 < p < +\infty$ then Calderón-Zygmund theory (Theorem 1.23) implies that $\Gamma * f$ is the derivative of a L^p function, so we may legitimately use the first Bernstein inequality to get

$$\big\|\chi(\lambda D)\big(\Gamma * f\big)\big\|_{L^p} = O\left(\frac{1}{\lambda^{1+d/p}}\right) \qquad \text{as } \lambda \to +\infty,$$

but this argument fails for the endpoint exponents, and in particular for $p = +\infty$ which corresponds to our framework for solutions. When $p = +\infty$, we may use the other properties of Singular Integral Operators described in Section 1.6 to see that the map $(-\Delta)^{-1}\partial_j\partial_k\partial_l : L^\infty \longrightarrow \mathrm{BMO}^{-1}$ is bounded. Since the space BMO^{-1} is a subspace of $\mathcal{S}'_h$ by Proposition 1.31, we may conclude that $\Gamma * f \in \mathcal{S}'_h$. However, we prefer giving an elementary proof based on an integral decomposition of the kernel Γ rather than resorting to Fefferman-Stein duality.

Proof of Proposition 2.24. We recall that the Fourier multiplier $\Delta_{-1} = \chi(D)$ is equal to the convolution operator $f \mapsto \psi * f$ (see Proposition 2.15). Therefore, we have, for any $\lambda > 0$,

$$\chi(\lambda D) = \psi_\lambda *, \qquad \text{where } \psi_\lambda(x) = \frac{1}{\lambda^d}\psi\left(\frac{x}{\lambda}\right).$$

Fix an exponent $\alpha > 0$ and define, for all $\lambda > 0$, the cut-off function $\theta_\lambda(x) = \chi(\lambda^{-\alpha}x)$. By proceeding as in the proof of Proposition 2.15, we have, for λ large enough that $\chi(\lambda D)\Delta_{-1} = \chi(\lambda D)$,

$$\chi(\lambda D)\Gamma = \psi_\lambda * \partial_j\partial_k\partial_l\big((1 - \theta_\lambda)E\big) + \partial_j\partial_k\partial_l\psi_\lambda * (\theta_\lambda E).$$

To prove the proposition, the Hausdorff-Young convolution inequality asserts that it is enough to show that the L^1 norm of both these convolution products is $O(\lambda^{-1}\log(\lambda))$. We start by studying the second one:

$$\|\partial_j\partial_k\partial_l\psi_\lambda * (\theta_\lambda E)\|_{L^1} \leq \|\partial_j\partial_k\partial_l\psi_\lambda\|_{L^1}\, \|\theta_\lambda E\|_{L^1}.$$

On the one hand, the three derivatives $\partial_j\partial_k\partial_l$ provide a λ^{-3} decay, as

$$\|\partial_j\partial_k\partial_l\psi_\lambda\|_{L^1} = \frac{1}{\lambda^3}\int \left|(\partial_j\partial_k\partial_l\psi)\left(\frac{x}{\lambda}\right)\right| \frac{\mathrm{d}x}{\lambda^d} = O\left(\frac{1}{\lambda^3}\right).$$

On the other hand, the fundamental solution $E(x)$ is locally integrable, because it has a $O(|x|^{d-2})$ singularity at $x=0$ when $d\geq 3$, and a $O(\log|x|)$ one if $d=2$. Since the function θ_λ is supported in the ball $B(0,2\lambda^\alpha)$, we have another inequality: in the case where $d\geq 3$, we get

$$\|\theta_\lambda E\|_{L^1} \leq C\int_{|x|\leq\lambda^\alpha} \frac{\mathrm{d}x}{|x|^{d-2}} = O(\lambda^{2\alpha}),$$

and in the case where $d=2$, we instead have

$$\begin{aligned}\|\theta_\lambda E\|_{L^1} &\leq C\int_{|x|\leq\lambda^\alpha} \log|x|\,\mathrm{d}x \leq C\int_0^{\lambda^\alpha} r\log(r)\mathrm{d}r \\ &= \frac{C}{2}r^2\Big(\log(r)-\frac{1}{2}\Big)\Big|_{r=0}^{\lambda^\alpha} = O\left(\lambda^{2\alpha}\log(\lambda)\right).\end{aligned}$$

We conclude that, in all dimensions $d\geq 2$, the convolution product $\partial_j\partial_k\partial_l\psi_\lambda * (\theta_\lambda E)$ has a L^1 norm that tends to zero as long as $\alpha<3/2$, at the speed $O(\lambda^{2\alpha-3}\log(\lambda))$.

We next look at the other convolution product $\psi_\lambda * \partial_j\partial_k\partial_l\big((1-\theta_\lambda)E\big)$. Here, we will take advantage of the integrability the third derivatives of $E(x)$ possess at $|x|\to+\infty$. However, this in itself is not enough, as we aim at showing decay as $\lambda\to+\infty$. We will also have to use the fact that the support of the cutoff $1-\theta_\lambda$ shrinks as λ becomes large. More precisely, we have the estimate, which holds in any dimension $d\geq 2$,

$$\left|\partial_j\partial_k\partial_l\big((1-\theta_\lambda)E\big)(x)\right| \leq C\frac{1-\mathbb{1}_{B(0,2\lambda^\alpha)}(x)}{|x|^{d+1}} := M_\lambda(x).$$

By using this on the convolution product, we find that

$$\left\|\psi_\lambda * \partial_j\partial_k\partial_l\big((1-\theta_\lambda)E\big)\right\|_{L^1} \leq C\|\psi_\lambda\|_{L^1}\|M_\lambda\|_{L^1} \leq C\|M_\lambda\|_{L^1}.$$

Finally, we may bound this last integral by

$$\|M_\lambda\|_{L^1} \leq C\int_{|x|\geq\lambda^\alpha}\frac{\mathrm{d}x}{|x|^{d+1}} = C\int_{\lambda^\alpha}^{+\infty}\frac{\mathrm{d}r}{r^2} = O\left(\frac{1}{\lambda^\alpha}\right).$$

Putting both estimates together, we have a low frequency inequality for the kernel Γ. In the limit $\lambda\to+\infty$,

$$\|\chi(\lambda D)\Gamma\|_{L^1} = O\left(\frac{1}{\lambda^\alpha}+\frac{1}{\lambda^{3-2\alpha}}\log(\lambda)\right),$$

and this gives convergence to 0 as long as $0<\alpha<3/2$. Taking $\alpha=1$ (the optimal value) ends proving our statement. □

Proposition 2.24 has the following consequence.

Corollary 2.27. *Let $f \in B^0_{\infty,\infty}(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$ be a field of matrices. Then we have*

$$\mathbb{P}\operatorname{div}(f) = \sum_{m\in\mathbb{Z}} \dot{\Delta}_m \mathbb{P}\operatorname{div}(f) \qquad \text{with convergence in } \mathcal{S}'.$$

We therefore define a bounded operator $\mathbb{P}\operatorname{div} : B^0_{\infty,\infty} \longrightarrow \mathfrak{B}^{-1}_{\infty,\infty} \subset \mathcal{S}'_h$. Here, $\mathfrak{B}^{-1}_{\infty,\infty}$ refers to the realization of $\dot{B}^{-1}_{\infty,\infty}$ as a subspace of $\mathcal{S}'_h$, see (1.10)*. The same statement holds for the operator $\mathbb{Q}\operatorname{div} = (\mathrm{Id} - \mathbb{P})\operatorname{div}$.*

Proof. Let A be the operator $A = \mathbb{P}\operatorname{div}$ or $A = \mathbb{Q}\operatorname{div}$. Because the symbol of A is a homogeneous function of degree one, Lemma 1.5 makes it clear that

$$\forall f \in B^0_{\infty,\infty}, \qquad \|Af\|_{\dot{B}^{-1}_{\infty,\infty}} \leq C\|f\|_{B^0_{\infty,\infty}}.$$

In addition, Proposition 2.24 shows that for any $f \in B^0_{\infty,\infty}$, the function $T_{j,k,l}f = (-\Delta)^{-1}\partial_j\partial_k\partial_l f$ lies in $\mathcal{S}'_h$, and the first Bernstein inequality (Lemma 1.7) that

$$\|\chi(\lambda D)\operatorname{div}(f)\|_{L^\infty} = \|\chi(\lambda D)\Delta_{-1}\operatorname{div}(f)\|_{L^\infty} = O\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda \to +\infty.$$

Therefore Af lies in $\mathcal{S}'_h$. $\square$

We now have all the necessary elements to prove Proposition 2.21.

Proof of Proposition 2.21. With Lemma 2.22 and Corollary 2.27 at our disposal, we are ready to complete the proof of Proposition 2.21, as it is equivalent for u to solve the projected problem (2.13) and for the polynomial ∇Q in Lemma 2.22 to be zero. Now, by Corollary 2.27, we see that $\partial_t u$ decomposes as

$$\partial_t u = -\mathbb{P}\operatorname{div}(u\otimes u) - \nabla Q \in \mathcal{S}'_h \oplus \mathbb{R}[X],$$

and so $\nabla Q \equiv 0$ if and only if $\partial_t u(t) \in \mathcal{S}'_h$ at all times $t \in]0, T[$.

We start by showing that assertions *(i)* and *(ii)* of Theorem 2.18 are equivalent: assume that condition *(ii)* holds, so that $u(t) - u(0) \in \mathcal{S}'_h$ for all $t \in [0, T[$. By differentiating with respect to time, we see that $\partial_t u(t)$ is also in $\mathcal{S}'_h$ for all $t \in]0, T[$. Indeed, since $u \in C^1_T(L^\infty)$ the difference quotients $h^{-1}(u(t+h) - u(t))$ will converge to its time derivative $\partial_t u(t)$ for the norm topology of L^∞, and the fact that the difference quotients are in $\mathcal{S}'_h \cap L^\infty$ insures that $\partial_t u(t)$ also is, because $\mathcal{S}'_h \cap L^\infty$ is a closed subspace of L^∞ (see Proposition 1.44). This implies that $\nabla Q \equiv 0$ and that u solves the projected problem (2.13) on $]0, T[\times\mathbb{R}^d$.

The fact that it does so with initial datum $u(0)$, and in the sense of Definition 2.17, simply stems from the C^1 regularity with respect to time.

Finally, we show that $V = u_0 - u(0)$ is a constant function. Any solution of the projected problem (2.13) must also be a solution of the Euler problem (2.10) with the same initial datum. Since u solves both problems (in the sense of Definitions 2.11 and 2.17), this implies that the weak form (2.12) of the momentum equation must hold for both initial data, so

$$\int \Big(u_0 - u(0)\Big) \cdot \phi \, \mathrm{d}x = 0$$

for all divergence-free $\phi \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^d)$, and so $V \in L^\infty$ is the gradient of some tempered distribution $V = \nabla h$. But since V is also divergence-free, h must be a harmonic polynomial, so that V is also polynomial. The fact that $V \in L^\infty$ forces V to be constant. We have shown *(ii)* to be true.

Next, we instead assume that condition *(i)* holds. By integrating with respect to time, we have

$$u(t) - u(0) = \int_0^t \partial_t u(s)\mathrm{d}s,$$

this integral being well defined as an element of the Banach space L^∞ (*e.g.* as a limit of Riemann sums). The fact that $\mathcal{S}'_h \cap L^\infty$ is closed in L^∞ for the strong topology implies that we must also have $u(t) - u(0) \in \mathcal{S}'_h$.

Condition *(iii)* implies *(ii)* because $\mathrm{BMO}^{-1} \subset \mathcal{S}'_h$. And if *(i)* is true, then the difference $u(t) - u(0)$ is given by

$$u(t) - u(0) = \int_0^t \partial_t u(s) \mathrm{d}s = - \int_0^t \mathbb{P} \operatorname{div}(u \otimes u) \mathrm{d}s.$$

Because $u \in C^0([0, T[; L^\infty)$, the function $\mathbb{P} \operatorname{div}(u \otimes u)$ lies in $C^0([0, T[; \mathrm{BMO}^{-1})$. Therefore the integrals above are well defined (as limits of Riemann sums) in the Banach space BMO^{-1}, thus implying *(iii)*.

We end by conditions *(iv)* and *(v)*. If *(i)* holds, then the pressure is given by Leray projection in the form of a singular integral operator

$$\pi = (-\Delta)^{-1} \partial_j \partial_k (u_j u_k) \qquad \text{in } C^0(]0, T[; \mathrm{BMO}).$$

In the above, time $t = 0$ is excluded: because the initial datum u_0 may be differ from the initial flow $u(0)$ by a constant, the pressure may exhibit a singularity at initial time

$$\nabla \pi = -\nabla (-\Delta)^{-1} \partial_j \partial_k (u_j u_k) + \delta_0 \otimes \big(u(0) - u_0\big).$$

At any rate, we deduce that both *(iv)* and *(v)* are true. Finally, assume *(v)*. Then, for every time $t \in]0, T[$, the derivative $\partial_t u(t)$ must lie in $\mathcal{S}'_h$. By the discussion above concerning points *(i)* and *(ii)*, it appears that the polynomial $\nabla Q(t)$ must be supported at time $t = 0$ and that the solution fulfills *(i)*. □

2.3.3 Equivalence of the Two Formulations: Low Time Regularity

In this paragraph, we fully prove Theorem 2.18 by working with the more general class of $C^0_T(L^\infty)$ solutions of (2.10). The obvious problem is that the derivative $\partial_t u$ might not exist as a function of time for these solutions. We will use a regularization procedure to circumvent this issue.

Proof of Theorem 2.18. Because we deal with solutions which have low time regularity, we take a mollification sequence $\big(K_\epsilon(t)\big)_{\epsilon > 0}$ such that K_1 is supported in the compact interval $[-1, 1]$, $\|K_1\|_{L^1} = 1$ and

$$K_\epsilon(t) = \frac{1}{\epsilon} K_1 \left(\frac{t}{\epsilon} \right).$$

The function K_1 is nonnegative by definition. Next, we extend all functions u and π to functions on the *open* set $t \in]-\infty, T[$ by setting them to zero on $]-\infty, 0[$. For the sake of simplicity, we continue to note u and π the extensions. This allows us to incorporate the initial data condition in the righthand side of the equations, and see that (u, π) solves

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla \pi = \delta_0(t) \otimes u_0(x) \qquad \text{in } \mathcal{D}'\big(]-\infty, T[\times \mathbb{R}^d\big).$$

For all $\epsilon > 0$, we note $(u_\epsilon, \pi_\epsilon) = K_\epsilon * (u, \pi)$, where the convolution is done with respect to the time variable $t \in]-\infty, T - \epsilon[$, as the kernel K_ϵ is supported in $[-\epsilon, \epsilon]$. We obtain, by convoluting with respect to time, an equality in the sense of $\mathcal{D}'(]-\infty, T - \epsilon[\times \mathbb{R}^d)$:

$$\begin{aligned} \partial_t u_\epsilon + K_\epsilon * \operatorname{div}(u \otimes u) + \nabla \pi_\epsilon &= (K_\epsilon * \delta_0)(t) \otimes u_0(x) \\ &= K_\epsilon(t) u_0(x). \end{aligned} \tag{2.18}$$

By taking the divergence of this equation and the m-th homogeneous dyadic block $\dot{\Delta}_m$, we see that

$$\forall m \in \mathbb{Z}, \qquad \dot{\Delta}_m \nabla \pi_\epsilon = \sum_{j,k} \dot{\Delta}_m \nabla(-\Delta)^{-1} \partial_j \partial_k K_\epsilon * (u_j u_k).$$

Because, for all $t \in]-\infty, T-\epsilon[$, the functions $K_\epsilon * (u_j u_k)(t)$ are bounded, Corollary 2.27 insures that we can sum the previous equation over $m \in \mathbb{Z}$ to get

$$\nabla \pi_\epsilon = \nabla Q_\epsilon(t) - K_\epsilon * (\mathrm{Id} - \mathbb{P}) \operatorname{div}(u \otimes u),$$

for some polynomial $Q_\epsilon(t) \in \mathbb{R}[X]$. By substituting this expression in (2.18), we find that

$$\partial_t u_\epsilon + K_\epsilon * \mathbb{P} \operatorname{div}(u \otimes u) + \nabla Q_\epsilon = K_\epsilon u_0, \qquad \text{for } t \in]-\infty, T-\epsilon[. \tag{2.19}$$

By taking the limit $\epsilon \to 0^+$ in the previous equation, we see that all terms converge in $\mathcal{D}'(]-\infty, T[\times \mathbb{R}^d)$ to some limit.[4] Therefore, the ∇Q_ϵ must also have a limit as $\epsilon \to 0^+$, which we will note ∇Q, and which satisfies

$$\partial_t u + \mathbb{P} \operatorname{div}(u \otimes u) + \nabla Q = \delta_0(t) \otimes u_0(x) \qquad \text{in } \mathcal{D}'\big(]-\infty, T[\times \mathbb{R}^d\big).$$

By taking the time convolution of this last equation by $K_\epsilon(t)$, we see that we in fact have $\nabla Q_\epsilon = K_\epsilon * \nabla Q$. The whole proof hinges on finding under what condition $\nabla Q = 0$ in $\mathcal{D}'(]0, T[\times \mathbb{R}^d)$.

Remark 2.28. In equation (2.19), the polynomial $Q_\epsilon(t) \in \mathbb{R}[X]$ is defined for every time $t \in]-\infty, T-\epsilon[$. This choice makes sense because all functions implied in (2.19) are C^∞ smooth with respect to time. However, Q_ϵ has no reason to define a distribution on $]-\infty, T-\epsilon[$. It is determined by π_ϵ (which is, by assumption, a distribution) *up to a constant summand* $C(t)$ which may be very singular, $C(t) = |t|^{-1}$ for example, or not even measurable.

By contrast, the derivative ∇Q_ϵ is a distribution on $]-\infty, T-\epsilon[$, and converges to a distribution ∇Q in the limit $\epsilon \to 0^+$. This convergence does not concern Q_ϵ, so Q is not *per se* a well defined object. But this poses no problem as we will only work with the (formal) derivative ∇Q.

We start by assuming that condition *(i)* holds, so that $\nabla Q = 0$ on $]0, T[\times \mathbb{R}^d$. This implies that $\nabla Q_\epsilon(t) = 0$ for $t \in [\epsilon, T-\epsilon]$, since K_ϵ is compactly supported in $[-\epsilon, \epsilon]$. Similarly, the term containing the initial datum u_0 in (2.18) vanishes as soon as $t \geq \epsilon$. Therefore, for $t \in [\epsilon, T-\epsilon]$,

$$\partial_t u_\epsilon(t) = -K_\epsilon * \mathbb{P} \operatorname{div}(u \otimes u) \in \mathcal{S}'_h.$$

Therefore, by integrating with respect to time, we find that

$$u_\epsilon(t) - u_\epsilon(\epsilon) = \int_\epsilon^t \partial_t u_\epsilon(s) \mathrm{d}s \in L^\infty \cap \mathcal{S}'_h.$$

Since $u_\epsilon \in C^\infty(L^\infty)$, this last integral is well-defined as an element of the Banach space $L^\infty \cap \mathcal{S}'_h$ (see Proposition 1.44) as, *e.g.* a limit of Riemann sums. We wish to take the limit $\epsilon \to 0^+$ in this last equation and use the fact that $L^\infty \cap \mathcal{S}'_h$ is closed in L^∞.

Thanks to the convergence $u_\epsilon \longrightarrow u$ in the space $C^0(]0, T[; L^\infty)$, which is equipped with the topology of uniform convergence on compact sets of $]0, T[$, we deduce on the one hand that we have pointwise convergence $u_\epsilon(t) \longrightarrow u(t)$. On the other hand, to compute the limit of $u_\epsilon(\epsilon)$, we write

$$\begin{aligned} \|u_\epsilon(\epsilon) - u(0)\|_{L^\infty} &\leq \|u_\epsilon(\epsilon) - u(\epsilon)\|_{L^\infty} + \|u(\epsilon) - u(0)\|_{L^\infty} \\ &= \|u_\epsilon(\epsilon) - u(\epsilon)\|_{L^\infty} + o(1). \end{aligned}$$

[4]For any fixed value of $\epsilon > 0$, the different terms do not really define a distribution on $]-\infty, T[\times \mathbb{R}^d$, but duality bracket involving a test function $\phi \in \mathcal{D}(]-\infty, T[\times \mathbb{R}^d)$ will always be defined for ϵ small enough.

Next, by writing explicitly the convolution integrals involved in $u_\epsilon(\epsilon) - u(\epsilon)$ and using the fact that the kernel K_ϵ is supported in $[-\epsilon, \epsilon]$, we get that

$$u_\epsilon(\epsilon) - u(\epsilon) = \int_0^{2\epsilon} K_\epsilon(\epsilon - s)u(s)\mathrm{d}s - u(\epsilon) = \int_0^{2\epsilon} K_\epsilon(\epsilon - s)\Big\{u(s) - u(\epsilon)\Big\}\mathrm{d}s,$$

The function $u \in C^0([0,T[; L^\infty)$ is uniformly continuous on any compact interval $[0,\eta]$. Therefore, we may take the L^∞ norm of the above equation to obtain

$$\|u_\epsilon(\epsilon) - u(\epsilon)\|_{L^\infty} \leq \|K_\epsilon\|_{L^1} \sup_{s\in[0,2\epsilon]} \|u(s) - u(\epsilon)\|_{L^\infty} \longrightarrow 0 \qquad \text{as } \epsilon \to 0^+.$$

This proves that $u_\epsilon(t) - u_\epsilon(\epsilon)$ converges to $u(t) - u(0)$ as $\epsilon \to 0^+$. Because $L^\infty \cap \mathcal{S}'_h$ is closed in L^∞, we finally deduce that $u(t) - u(0) \in \mathcal{S}'_h$ for all $0 \leq t < T$.

Now, let us assume that $u(t) - u(0) \in \mathcal{S}'_h$ for all $t \in [0,T[$. We take the convolution of $u(t) - u(0)$ with K_ϵ to find that $u_\epsilon(t) - u(0)$ also lies in $L^\infty \cap \mathcal{S}'_h$, but only for $t \in]\epsilon, T-\epsilon[$, as the condition $u(t) - u(0) \in \mathcal{S}'_h$ does not hold on the extension for $t \notin [0,T[$, as $u(0)$ need not be in $\mathcal{S}'_h$. Since $u_\epsilon(t) - u(0)$ is a C^1 function with respect to time, we can differentiate. The derivative $\partial_t u_\epsilon$ will then be found as the L^∞ limit of the difference quotients, and will therefore also be an element of $L^\infty \cap \mathcal{S}'_h$,

$$\forall t \in]\epsilon, T-\epsilon[, \qquad \partial_t u_\epsilon(t) \in \mathcal{S}'_h.$$

This implies that the polynomial $\nabla Q_\epsilon(t)$ from (2.19) must also be in $\mathcal{S}'_h$ if $t \in]\epsilon, T-\epsilon[$, and so must be zero on that time interval. Therefore, $\nabla Q \equiv 0$ in $\mathcal{D}'(]0,T[\times\mathbb{R}^d)$. This means that u solves the projected system (2.13), we just have to check the assertion concerning the initial datum for the velocity.

Since the function u solves the projected equation, we may use it to prove that u is in fact $C^1_T(\dot{B}^{-1}_{\infty,\infty})$ regular. The C^1 time regularity implies that u will be a weak solution of the projected equation with initial datum $u(0)$, as in Definition 2.17.

Lastly, since u solves the projected problem with initial datum $u(0)$, it must solve the Euler system with the datum $u(0)$. Therefore, u is a weak solution of the Euler equations, as in Definition 2.11 with both initial data u_0 and $u(0)$, and so $u_0 - u(0)$ must be a constant function, as shown in Subsection 2.3.2 above. This shows the equivalence of points *(i)* and *(ii)*.

Now that we have established that *(i)* and *(ii)* are equivalent, we may focus on the last three points. Of course, *(iii)* implies *(ii)*. And if u is a solution of the projected problem, so that *(i)* holds, we must have

$$\partial_t u = -\mathbb{P}\,\mathrm{div}(u \otimes u) \qquad \text{in } \mathcal{D}'(]0,T[\times\mathbb{R}^d).$$

This implies that $\partial_t u$ is uniformly continuous with respect to $t \in]0,T[$ in the BMO^{-1} topology, so that for any $t \in [\eta, T[$ the following integral is well defined:

$$u(t) - u(\eta) = -\int_\eta^t \mathbb{P}\,\mathrm{div}(u\otimes u)\mathrm{d}s \underset{\eta\to 0^+}{\longrightarrow} -\int_0^t \mathbb{P}\,\mathrm{div}(u\otimes u)\mathrm{d}s \qquad \text{in } \mathrm{BMO}^{-1}.$$

Because $u \in C^0([0,T[; L^\infty)$, the lefthand side of this equation converges (in L^∞) to $u(t) - u(0)$ as $\eta \to 0^+$. Uniqueness of the limit implies that *(iii)* holds.

Finally, if u is a solution of the projected problem (that is, if *(i)* is true), then we may conclude to *(iv)* and *(v)* exactly as in the proof of Proposition 2.21. On the other hand, if *(v)* is true, then we check that $\nabla\pi_\epsilon(t) \in \mathcal{S}'_h$. For any test function $\phi \in \mathcal{S}$ and any $t \in]\epsilon, T-\epsilon[$,

$$\big\langle \chi(\lambda D)\nabla\pi_\epsilon, \phi\big\rangle_{\mathcal{S}'\times\mathcal{S}} = \int K_\epsilon(t-s)\big\langle\chi(\lambda D)\nabla\pi(s), \phi\big\rangle_{\mathcal{S}'\times\mathcal{S}}\,\mathrm{d}s.$$

Because $\nabla\pi \in C^0(]0,T[;\mathcal{S}')$, the bracket in the integral is a continuous function of time that converges to zero as $\lambda \to +\infty$ (by assumption) while remaining uniformly bounded. Therefore, dominated convergence guarantees that the whole integral tends to zero as $\lambda \to +\infty$, hence $\nabla\pi_\epsilon(t) \in \mathcal{S}'_h$. With that remark in mind, bringing the proof to its conclusion is only a matter of noticing that

$$\nabla\pi_\epsilon = K_\epsilon * \nabla(-\Delta)^{-1}\partial_j\partial_k(u_ju_k) + \nabla Q_\epsilon \in \mathcal{S}'_h \oplus \mathbb{R}[X].$$

Since $\nabla\pi_\epsilon \in \mathcal{S}'_h$, the polynomial must be zero $\nabla Q_\epsilon = 0$ and we deduce that $\nabla Q = 0$ on $]0,T[\times\mathbb{R}^d$. □

2.4 Two Applications

In this short paragraph, we provide two applications of Theorem 2.18 to, firstly, a well-posedness result in Besov spaces and, secondly, Serfati solutions of the Euler system.

2.4.1 A Full Well-Posedness Result

Theorem 2.18 above provides us with a full well-posedness result for the Euler system in the space $C^0_T(B^1_{\infty,1})$.

Corollary 2.29. *Consider $u_0 \in B^1_{\infty,1}$ a divergence-free initial datum. There exists a time $T > 0$ such that the Euler problem (2.10) has a unique solution $u \in C^0([0,T[;B^1_{\infty,1})$ (in the sense of Definition 2.11) related to the initial datum u_0 that satisfies*

$$u(0) = u_0 \qquad \text{and} \qquad u(t) - u(0) \in \mathcal{S}'_h \text{ for } t \in [0,T[. \tag{2.20}$$

Moreover, this solutions lies in the space $C^1([0,T[;B^0_{\infty,1})$.

Remark 2.30. Note that condition (2.20) is a rather natural one for a well-posedness result, as $u(t) - u(0) \in \mathcal{S}'_h$ implies that the flow is, unlike the Poiseuille-type flow (2.2), not driven by an exterior action (recall that $\mathcal{S}'_h$ is a space of functions that are "on average" zero at $|x| \to +\infty$). A flow that is left to its own devices must be deterministic.

Proof. Thanks to Theorem 2.18, the proof is straightforward. With the help of the results of [87], we have a $T > 0$ such that the projected problem has a unique solution $u \in C^0_T(B^1_{\infty,1}) \cap C^1_T(B^0_{\infty,1})$ with initial datum u_0. This solution must also solve the original Euler problem with the same initial datum, as well as fulfilling (2.20).

On the other hand, by Theorem 2.18, any solution $v \in C^0_T(B^1_{\infty,1}) \hookrightarrow C^0_T(L^\infty)$ that fulfills condition (2.20) must also solve the projected problem (2.13) with the initial datum u_0, and so must coincide with u on $[0,T[\times\mathbb{R}^d$. □

2.4.2 Serfati Solutions

In this paragraph, we work exclusively on the 2D plane $\mathbb{R}^2$, so that $d = 2$.

Serfati solutions of the Euler equations[5] are a class of 2D bounded solutions introduced by Philippe Serfati in [97] (see also [3] for an english presentation). These solutions have a threefold description: a solution $u \in C^0(\mathbb{R}_+;L^\infty(\mathbb{R}^2))$ is a *Serfati solution* if

(i) both u and the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ are uniformly bounded

$$\sup_{t\geq 0}\Big(\|u\|_{L^\infty} + \|\omega\|_{L^\infty}\Big) < +\infty, \tag{2.21}$$

[5] Many thanks are due to Raphaël Danchin for introducing us to this topic and to Dragoş Iftimie for asking the questions studied in this paragraph.

(ii) the vorticity solves the usual pure transport equation $\partial_t\omega + u\cdot\nabla\omega = 0$ in the sense of distributions,

(iii) the velocity u satisfies the *Serfati identity*: for any radial cut-off function $\theta \in \mathcal{D}$ such that $\theta(x) = 1$ around $x = 0$, we have

$$u(t) - u(0) = (\theta K) * \big(\omega(t) - \omega(0)\big) - \int_0^t \nabla\nabla^\perp\big((1-\theta)K\big) * (u \otimes u)\mathrm{d}s, \tag{2.22}$$

where in the equation above $K(x) = x^\perp/|x|^2$ is the Biot-Savart kernel and the convolution product contains an implicit contraction between both 2×2 matrices $\nabla\nabla^\perp\big((1-\theta)K\big)$ and $u\otimes u$.

The Serfati identity follows naturally from the Euler equations in their vorticity form. Formally, the velocity flow satisfies $u = K * \omega$ and hence

$$\partial_t u = (\theta K) * \partial_t\omega + \big((1-\theta)K\big) * \partial_t\omega.$$

On the one hand, the first convolution product $(\theta K) * \omega$ makes sense if $\omega \in L^\infty$ because the Biot-Savart kernel is locally integrable. Integration with respect to time gives the first term in the righthandside of (2.22). On the other hand, by substituting $\partial_t\omega = -u\cdot\nabla\omega = \mathrm{curl}\ \mathrm{div}(u\otimes u)$ and integrating by parts, we obtain the second term:

$$\int_0^t \big((1-\theta)K\big) * \partial_t\omega\mathrm{d}s = -\int_0^t \nabla\nabla^\perp\big((1-\theta)K\big) * (u\otimes u)\mathrm{d}s.$$

A notable feature of Serfati solutions is that they lead to *global a priori estimates* for the quantity $\|u\|_{L^\infty} + \|\omega\|_{L^\infty}$ and to a well-posedness result: for any $u_0 \in L^\infty$ with $\omega_0 = \mathrm{curl}\,(u_0) \in L^\infty$, there exists a unique Serfati solution with initial datum u_0.

As noted in Section 7 of [3], this means that requiring a solution of the Euler system to fulfill the Serfati identity rules out the pathological solutions, such as (2.2), that create non-uniqueness. In fact, as is deeply discussed in [68], a (weak) solution of the Euler system (2.10) which satisfies (2.21) is uniquely determined by the initial value and an asymptotic function $U_\infty(t)$ if one of the following sufficient conditions is satisfied:

(i) a "generalized" Serfati identity holds, where the uniform flow $U_\infty(t)$ is added to (2.22),

(ii) the velocity satisfies a renormalized Biot-Savart law: if θ is as above and $\theta_R(x) = \theta(x/R)$, then

$$(\theta_R K) * \big(\omega(t) - \omega(0)\big) \underset{R\to+\infty}{\longrightarrow} U_\infty(t) + u(t) - u(0)$$

locally uniformly in the sense of the norm (2.21),

(iii) the pressure $\pi(t,x)$ satisfies

$$\pi(t,x) - U_\infty'(t)\cdot x \ \in L^\infty_T(\mathrm{BMO}).$$

Here, we show the relation between Serfati solutions and our own result: more precisely, we prove that a Serfati solution fulfills the equivalent conditions of Theorem 2.18, and hence is a solution of the projected problem.

Theorem 2.31. *Let $u \in C^0(\mathbb{R}_+; L^\infty)$ be a Serfati solution of the Euler system. Then, for all $\alpha < 1$,*

$$\big\|\chi(\lambda D)\big(u(t) - u(0)\big)\big\|_{L^\infty} = O\left(\frac{1}{\lambda^\alpha}\right) \qquad \text{as } \lambda \to +\infty.$$

In particular, u is a solution of the projected problem (2.13) with initial value $u(0)$.

Remark 2.32. Serfati solutions are more general than the $B^1_{\infty,1}$ solutions of Pak and Park [87], because $\omega \in L^\infty$ does not imply that $\nabla u \in L^\infty$, as would be the case if we had $u \in B^1_{\infty,1}$. On the other hand, Serfati solutions form a more restrictive framework for the Euler equations than our simple $C^0(L^\infty)$ context: firstly, Serfati solutions are only defined in 2D, and secondly Theorem 2.18 holds for very rough solutions, whereas the Serfati identity requires $\omega \in L^\infty$ to be written.

Proof. Let, as usual, $\psi_\lambda \in \mathcal{S}$ such that $\widehat{\psi_\lambda}(\xi) = \chi(\lambda\xi)$ and consider the convolution product of the Serfati identity (2.22)

$$\psi_\lambda * \big(u(t) - u(0)\big) = \psi_\lambda * (\theta K) * \big(\omega(t) - \omega(0)\big) - \int_0^t \psi_\lambda * \nabla\nabla^\perp\big((1-\theta)K\big) * (u \otimes u)\mathrm{d}s.$$

This equation is true for all nonnegative radial $\theta \in \mathcal{D}$ such that $\theta(x) = 1$ near $x = 0$. As in the proof of Proposition 2.24, our aim is to estimate the convolution products with the Hausdorff convolution inequality to obtain our result. We focus on the first term. Integration by parts gives

$$\begin{aligned} \Big\|\psi_\lambda * (\theta K) * \big(\omega(t) - \omega(0)\big)\Big\|_{L^\infty} &= \Big\|\nabla^\perp\psi_\lambda * (\theta K) * \big(u(t) - u(0)\big)\Big\|_{L^\infty} \\ &\le \|\nabla\psi_\lambda\|_{L^1}\|\theta K\|_{L^1}\|u(t) - u(0)\|_{L^\infty} \\ &\le \frac{1}{\lambda}\|\nabla\psi\|_{L^1}\|\theta K\|_{L^1}\|u(t) - u(0)\|_{L^\infty}. \end{aligned} \tag{2.23}$$

We leave here, for now, this first term and come to the second one. Similarly, the Hausdorff convolution inequality gives

$$\begin{aligned} \Big\|\psi_\lambda * \nabla\nabla^\perp\big((1-\theta)K\big) * (u\otimes u)\Big\|_{L^\infty} &\le \|\psi_\lambda\|_{L^1}\big\|\nabla^2\big((1-\theta)K\big)\big\|_{L^1}\|u\|^2_{L^\infty} \\ &\le \big\|\nabla^2\big((1-\theta)K\big)\big\|_{L^1}\|\psi\|_{L^1}\|u\|^2_{L^\infty}, \end{aligned}$$

but this is not sufficient: the upper bound we have found does not depend on λ, so it will not decay as $\lambda \to +\infty$. To force this upper bound to be small, we take advantage of the fact that the Serfati identity holds for all radial cut-off functions $\theta \in \mathcal{D}$. Fix a such $\theta \in \mathcal{D}$ that is supported in the ball $|x| \le 1$ and define $\theta_\lambda(x) = \theta(x/\lambda^\alpha)$ for some $\alpha > 0$ to be fixed later. Then we have

$$\big\|\nabla^2\big((1-\theta)K\big)\big\|_{L^1} \le C\int_{\lambda^\alpha}^{+\infty}\frac{\mathrm{d}r}{r^2} = O\left(\frac{1}{\lambda^\alpha}\right) \qquad \text{as } \lambda \to +\infty,$$

and therefore

$$\left\|\int_0^t \psi_\lambda * \nabla\nabla^\perp\big((1-\theta)K\big) * (u\otimes u)\mathrm{d}s\right\|_{L^\infty} = O\left(\frac{1}{\lambda^\alpha}\right) \qquad \text{as } \lambda \to +\infty. \tag{2.24}$$

On the other hand, our choice of cut-off function comes at a cost: because the support of θ_λ grows as λ does, we lose decay in (2.23):

$$\frac{1}{\lambda}\|\theta_\lambda K\|_{L^1} = O\left(\frac{1}{\lambda^{1-\alpha}}\right) \qquad \text{as } \lambda \to +\infty.$$

By putting this together with (2.24) and choosing the optimal $\alpha = 1/2$, we see that $\chi(\lambda D)(u(t) - u(0))$ does indeed tend to zero in L^∞ at the speed $O(\lambda^{-1/2})$. With Theorem 2.18, this is enough to prove that u is a solution of the projected problem.

However, we are not entirely sated, because the $O(\lambda^{-1/2})$ does not seem optimal: by Proposition 2.24, we know that the decay is truly of order roughly $O(\lambda^{-1})$. By revisiting the computations above, we see that the estimates in (2.23) are not really optimal: because convolution by ψ_λ is a low frequency cut-off operator $\chi(\lambda D)$, we have

$$\begin{aligned} \psi_\lambda * (\theta K) * \big(\omega(t) - \omega(0)\big) &= \psi_{2\lambda} * \psi_\lambda * (\theta K) * \big(\omega(t) - \omega(0)\big) \\ &= \nabla\psi_\lambda * (\theta_\lambda K) * \Big(\psi_{2\lambda} * (u(t) - u(0))\Big). \end{aligned}$$

Now, thanks to the estimates we have just found, we know that $\psi_{2\lambda} * (u(t) - u(0))$ also decays in L^∞ at the speed $O(\lambda^{-1/2})$. By taking this into account, we obtain

$$\left\| \psi_\lambda * (\theta K) * \big(\omega(t) - \omega(0)\big) \right\|_{L^\infty} = O\left(\frac{1}{\lambda^{3/2-\alpha}} \right) \qquad \text{as } \lambda \to +\infty$$

and by setting $\alpha = 3/4$ we obtain a $O(\lambda^{-3/4})$ estimate instead of a $O(\lambda^{-1/2})$ one, and we can once again use this improved bound in (2.23)... By iterating, we find that the bound may be brought to $O(\lambda^{-\alpha})$ for any $\alpha < 1$, but at the price of a constant that depends on α. $\square$

Chapter 3

Elsässer Variables and Local Solutions

Whenever I hear the word "magnetohydrodynamic"
my brain just replaces it with "magic".
Randall Munroe[1]

3.1 Introduction

With this chapter, we begin our study of magnetohydrodynamics, and in particular the question of existence and uniqueness of solutions to the Cauchy problem. As we have seen in the introduction, the structure of the equations implies that finding any kind of global solution for arbitrary initial data is challenging in the extreme. Nevertheless, it is possible to use fixed point-type arguments to construct regular solutions on a finite time interval.

Such a construction is the main topic in this chapter. We will then comment abundantly on the lifespan of solutions, providing various continuation criteria and lower bounds for the lifespan in regimes of low magnetic fields. This chapter is a combination of our published work in [28], [30] and [25].

3.1.1 Conservation of Energy and Symmetric Structure of the MHD System

Recall the ideal MHD system: a generalization of the Euler equations involving a self induced magnetic field $b(t,x) \in \mathbb{R}^d$ which reads

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \pi = (b \cdot \nabla) b \\ \partial_t b + (u \cdot \nabla) b = (b \cdot \nabla) u \\ \operatorname{div}(u) = 0. \end{cases} \tag{3.1}$$

In the equations above, $\pi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is the MHD pressure, which is the sum of the usual hydrodynamic pressure and the magnetic pressure $\frac{1}{2}|b|^2$. The difficulty of finding *a priori* bounds for these equations is obvious at first glance: both equations for u and b are transport equations (ignoring the non-local MHD pressure term) with righthand sides containing derivatives of u and b. Because of the propagation of singularities in the flow of transport equations, any solution (u,b) of (3.1) is expected to have the same regularity as $(b \cdot \nabla)b$ and $(b \cdot \nabla)u$, that is to say less regularity than the solution (u,b) itself!

However, if this naive approach may worry us that solutions instantaneously degenerate, the well-posedness theory of (3.1) is saved by the underlying structure of the equations.

[1] Randall Munroe, *xkcd*, `https://xkcd.com/1851`. As puts Munroe in the same comic, *Magnetohydrodynamics combines the intuitive nature of Maxwell's equations with the easy solvability of the Navier-Stokes equations. It's so straightforward physicists add "relativistic" or "quantum" just to keep it from getting boring.*

Firstly, it is notable that total energy (kinetic and magnetic) is conserved in the equations: taking the scalar product of the momentum equation with u and of the magnetic field equation with b and integrating yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(|u|^2 + |b|^2\right) \mathrm{d}x = 0.$$

We refer to the introduction of the dissertation for a much more detailed discussion of energy conservation laws and their symmetries. Our point here is that (3.1) essentially has the structure of a quasi-linear symmetric hyperbolic system (see Chapter 4 in [7] for a general theory of these equations). In other words, conservation of energy implies that the L^2 norms of the derivatives of u and b solve a differential inequality. By differentiating (3.1), we see that the derivatives $\nabla^k u$ and $\nabla^k b$ solve (3.1) up to commutator terms, so that, schematically speaking,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(|\nabla^k u|^2 + |\nabla^k b|^2\right) \mathrm{d}x = \text{commutator terms.}$$

As we will see, commutator terms are typically better behaved than the functions they involve. The ones above can be estimated in Sobolev H^s norms for $s > 1 + d/2$. All in all, it is possible to prove existence and uniqueness of local solutions in subcritical H^s Sobolev spaces $s > 1 + d/2$.

Theorem 3.1 (Schmidt, 1987, [95]). *Let $u_0, b_0 \in H^s$ be divergence free initial data with $s > 1+d/2$. Then there exists a $T > 0$ and a unique solution $(u, b) \in C^0([0, T[; H^s)$ of (3.1) associated to the initial data (u_0, b_0).*

Remark 3.2. The reader will have noted we assert that standard hyperbolic theory can be used to solve the ideal MHD equations while (3.1) is not *per se* hyperbolic: the MHD pressure $\nabla\pi$ is in fact a non-local function of u and b. However, we will see that the presence of $\nabla\pi$ creates no additional difficulty. Although (3.1) will not inherit all the properties of hyperbolic systems, such as the finite propagation speed of information, it behaves sufficiently like a hyperbolic system for us to apply the general methods of the theory: Besov-Lipschitz solutions, commutator estimates, Beale-Kato-Majda like continuation criteria, *etc.*

3.1.2 Introducing Elsässer Variables

The major shortcoming of hyperbolic theory is that it relies on energy estimates, and therefore cannot hold in L^p-based spaces for $p \neq 2$, for instance in other $W^{s,p}$ Sobolev spaces. In fact, it is a known feature of symmetric hyperbolic systems that they are ill-posed for $p \neq 2$ unless all the equations decouple into transport equations (see for instance the work of Brenner [15]).

The miracle lies in that a similar kind of decoupling exists for the MHD systems. It is a simple change of variables introduced in 1950 by the German-American Physicist W. M. Elsässer [41] which rises naturally from the divergence form of the equations:

$$\begin{aligned}
&\partial_t u + \mathrm{div}(u \otimes u - b \otimes b) + \nabla\pi = 0\\
&\partial_t b + \mathrm{div}(u \otimes b\ - b \otimes u) = 0.
\end{aligned}$$

Note how the tensor difference $u \otimes u - b \otimes b$ appearing in the momentum equation ressembles the identity $x^2 - y^2 = (x + y)(x - y)$. Inspired by this remark, we define the *Elsässer variables*

$$\alpha = u + b \qquad \text{and} \qquad \beta = u - b,$$

whose products are

$$\begin{aligned}
\alpha \otimes \beta &= u \otimes u - b \otimes b + b \otimes u - u \otimes b\\
\beta \otimes \alpha &= u \otimes u - b \otimes b + u \otimes b - b \otimes u.
\end{aligned}$$

Therefore, taking the sum and the difference of both momentum and magnetic field equations in (3.1) gives a new system, which we will refer to as the *Elsässer system*

$$
\begin{cases}
\partial_t \alpha + (\beta \cdot \nabla)\alpha + \nabla \pi_1 = 0 \\
\partial_t \beta + (\alpha \cdot \nabla)\beta + \nabla \pi_2 = 0 \\
\mathrm{div}(\alpha) = \mathrm{div}(\beta) = 0,
\end{cases} \tag{3.2}
$$

where π_1 and π_2 are two *independent* variables. Though this may seem surprising (taking the sum and difference of the u and b equations gives $\nabla\pi_1 = \nabla\pi_2 = \nabla\pi$), we will explain thoroughly below our choice of introducing a new apparently artificial variable.

Here, many things must be noted. Firstly, we have every reason to be satisfied with the new equations (3.2), as they are indeed a system of transport equations up to the non-local pressure terms. As with transport equations, we have high hopes to solve (3.2) in L^p-based spaces for the whole range of $p \in [1, +\infty]$.

Secondly, we must ponder on the precise relation between both systems (3.1) and (3.2). Note that in the ideal MHD equations, as they are written in (3.1) above, there are $2d + 1$ equations[2] for $2d + 1$ unknowns (u, b, π), while the Elsässer system (3.2) has $2d + 2$ equations for $2d + 2$ unknowns $(\alpha, \beta, \pi_1, \pi_2)$. The reason for this difference is twofold (see the introduction of [95]). On the one hand, having two *a priori independent* pressure functions π_1 and π_2 is necessary to keep the system from being formally underdetermined: both terms are needed to enforce the two *independent* divergence constraints $\mathrm{div}(\alpha) = \mathrm{div}(\beta) = 0$. On the other hand, the structure of the equations implies that we must in fact have, under fairly general assumptions which we will explicit in Theorem 3.11 below, $\nabla\pi_1 = \nabla\pi_2 = \nabla\pi$ as both functions solve the same elliptic problem

$$-\Delta\pi_1 = \partial_j\partial_k(\beta_k\alpha_j) = \partial_j\partial_k(\alpha_k\beta_j) = -\Delta\pi_2.$$

Therefore, if indeed $\nabla\pi = \nabla\pi_1 = \nabla\pi_2$, taking the sum and the difference of both equations on α and β lets us recover the original MHD system with, on one side the momentum equation for $u = \frac{1}{2}(\alpha + \beta)$

$$\partial_t u + (u \cdot \nabla)u - (b \cdot \nabla)b = -\frac{1}{2}\nabla(\pi_1 + \pi_2) = -\nabla\pi,$$

and the magnetic field equation $b = \frac{1}{2}(\alpha - \beta)$

$$\partial_t b + (u \cdot \nabla)b - (b \cdot \nabla)u = \frac{1}{2}\nabla(\pi_2 - \pi_1) = 0.$$

The underlying structure of the ideal MHD system, with "decoupled" variables and (non-local) transport equations makes it possible to solve the initial value problem in spaces with $p \neq 2$ as did Alekseev (1982, [2]) and Secchi (1983, [96]). Further down the line, Miao and Yuan (2006, [83]) show existence and uniqueness in critical Besov-Lipschitz spaces.

Theorem 3.3 (Miao, Yuan, 2006, [83])**.** *Let $p \in [1, +\infty]$ and $u_0, b_0 \in B^{1+d/p}_{p,1}$ be two divergence free functions. There exists a time $T > 0$ with*

$$T \geq \frac{C}{\|u_0, b_0\|_{B^{1+d/p}_{p,1}}}$$

such that system (3.1) *has a unique solution $(u, b) \in C^0([0, T[; B^{1+d/p}_{p,1})$ provided*[3] *the MHD pressure π is given by $\pi = (-\Delta)^{-1}\partial_j\partial_k(u_ju_k - b_jb_k)$.*

[2] As explained in the introduction, the divergence free condition $\mathrm{div}(b) = 0$ is not needed to determine the dynamics of the system. Provided b is initially divergence-free, the magnetic field naturally stays so at all times.

[3] This extra assumption is entirely implicit in [83]. It is nonetheless necessary to provide uniqueness of solutions in the class $C^0(B^1_\infty)$, as we have abundantly explained in the previous chapter in the special case of the Euler equations, that is of (3.1) with $b \equiv 0$.

Remark 3.4. Well-posedness has also been obtained for the ideal incompressible MHD system with variable density. In that case, the magnetic field equation is unchanged, but the momentum equation has to be modified to account for density changes

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u - b \otimes b) + \nabla \pi = 0.$$

The addition of the density heavily affects the structure of the system: the Elsässer variables $\alpha, \beta := u \pm b/\sqrt{\rho}$ no longer solve transport-like equations. Therefore, well-posedness can only be acquired by using energy conservation and the hyperbolic properties of the equations. This is reflected by the fact that the only results so far are obtained in supercritical Sobolev spaces H^s (Secchi, 1993, [96] for small initial density and Fan-Zhou, 2010, [105] or He-Fan-Zhou, 2016, [62] for general densities).

Our goal in this chapter is to improve Theorem 3.3 in four ways.

(i) First and foremost, we will be essentially interested in bounded (Besov-Lipschitz) solutions: for such solutions, it is not at all obvious that the pressure is given by the singular integral $\pi = (-\Delta)^{-1}\partial_j\partial_k(u_j u_k - b_j b_k)$. In fact, for bounded (regular) solutions, it is not true that the original MHD system (3.1) is equivalent to the Elsässer system (3.2). Our first goal will be to use the methods of the previous chapter in order to clarify the precise relations between (3.1) and (3.2) and give a complete well-posedness result, analogous to Corollary 2.29. This is the purpose of Theorems 3.11 and 3.12.

(ii) Secondly, the proof of [83] is based on compositions of the solutions by the flows of α and β, thus making the computations highly technical. We will make maximum use of the Elsässer variables and the simplifications they provide so as to obtain a simpler proof.

(iii) Thirdly, we will examine continuation criteria in the spirit of Beale-Kato-Majda (BKM below) [8]. As proved by Caflish, Klapper and Steele (1997, [16]), a regular solution (u, b) may be extended beyond time T if and only if

$$\int_0^T \Big(\|\operatorname{curl}(u)\|_{L^\infty} + \|\operatorname{curl}(b)\|_{L^\infty} \Big)\, \mathrm{d}t < +\infty.$$

Several improvements have been achieved in this direction: [104] allows the L^∞ norms to be replaced by the Besov norms $\dot{B}^0_{\infty,\infty}$, and this result was in turn improved by [18]. The L^∞ norms can also be replaced by the Triebel-Lizorkin norms $\dot{F}^0_{\infty,\infty}$, as shown in [23]. All these results have in common that they need both the velocity and the magnetic fields to have bounded curls. We will use the structure of the equations to uncover continuation criteria which involve *only one* vector quantity.

(iv) Finally, we wish to improve the lower bound for the lifespan of solutions provided in Theorem 3.3. Solutions of the 2D Euler equations are known to be global, so that the lifespan of plane MHD solutions should increase to $T_{\text{EULER}} = +\infty$ in the regime of low magnetic fields $b_0 \to 0$. Our last goal in this chapter will be to find a lower bound which accounts for this: we will show that the lifespan T of a solution with initial data (u_0, b_0) is higher than (see Theorem 3.29)

$$T \geq \frac{C}{\|u_0\|_{L^2 \cap B^1_{\infty,1}}} \log\left\{ 1 + \log\left[1 + \log\left(1 + C\frac{\|u_0\|_{L^2 \cap B^2_{\infty,1}}}{\|b_0\|_{B^1_{\infty,1}}} \right)\right]\right\}$$

In other words, we have some sort of "asymptotic well-posedness" of the 2D ideal MHD equations (3.1) in the regime of small magnetic fields. Such improved lower bounds are very much in the spirit of that obtained by Danchin-Fanelli [34] or the recent paper of Sbaiz [94] for the non-homogeneous incompressible Euler equations in the limit of small variations of density or Fanelli and Liao in [43] for a zero Mach number limit system.

As we are going to see, these bounds do not rely on a perturbative argument around special equilibria, like *e.g.* [9] and [39] for Euler flows and porous media respectively; yet, one has to remark that, contrarily to our result, those results are global.

3.2 Elsässer Variables

The purpose of the Section is to study the equivalence between the Elsässer and the classical MHD systems in the framework of bounded weak solutions as an application of the methods developed in Chapter 2.

Our starting point here is, as the reader may have gathered from the discussion above, that the relation between the physical and the Elsässer variables is unclear: while it is always true that any solution (u, b) of the classical MHD system (3.1) provides a solution (α, β) of the Elsässer system (3.2), the converse is not true if (α, β) do not possess some kind of decay at infinity.

As we have explained above, if (α, β) is a solution of (3.2) with pressures π_1 and π_2, then by setting $(u, b) = \frac{1}{2}(\alpha + \beta, \alpha - \beta)$ we construct a solution u of the momentum equation with MHD pressure $\pi = \frac{1}{2}(\pi_1 + \pi_2)$ and a solution b of

$$\partial_t b + \operatorname{div}(u \otimes b - b \otimes u) = \frac{1}{2}\nabla(\pi_2 - \pi_1) \tag{3.3}$$

so that b solves the magnetic field equation if and only if $\nabla\pi_1 = \nabla\pi_2$. Of course, there is not much possibility for this equality not holding: by taking the divergence of (3.3), we see that the difference $Q = \pi_2 - \pi_1$ is a harmonic polynomial

$$\frac{1}{2}\Delta(\pi_2 - \pi_1) = \partial_j\partial_k(u_j b_k - u_k b_j) = 0,$$

but this does not imply that $Q = 0$. Let us give an example inspired by the uniform flow (2.2) of the previous Chapter: consider

$$\alpha(t, x) = f(t) \qquad \text{and} \qquad \beta(t, x) = -f(t), \tag{3.4}$$

where $f \in C^\infty(\mathbb{R}; \mathbb{R}^d)$ a smooth function. Then (α, β) solves the Elsässer system with pressure functions $\pi_1(t, x) = -f'(t) \cdot x = -\pi_2(t, x)$ but we have $\frac{1}{2}\nabla(\pi_2 - \pi_1) = f'(t) \neq 0$. We see that we have the same kind of issue here we had with the Euler system in Chapter 2.

Remark 3.5. The solution (3.4) displayed above offers an interesting discussion as to whether it is physically relevant. Written in physical variables, it reads

$$u(t, x) = 0 \qquad \text{and} \qquad b(t, x) = 2f(t). \tag{3.5}$$

On the one hand, it would seem that this solution has an obvious physical interpretation: a uniform magnetic field may be created in the interior of an infinite ideal cylindrical solenoid, so that (3.5) could be seen as generated by solenoidal-type currents "at infinity".

However, it turns out that this is an acceptable solution of the Maxwell equations (in the magnetic limit, see the discussion at p. 28 in the Introduction) only for non-variable fields $f'(t) = 0$. Because time variations of the magnetic field induce a nonzero electromotive force (by Faraday's law), the presence of an electric field e only is possible if the fluid is in motion, as $e = -u \times b$.

Remark 3.6. In Chapter 2, any solution of the Euler equation could be obtained from a solution of the projected system through a "generalized Galilean transform". One may wonder if a similar process would allow any solution of the Elsässer system to be obtained from the MHD equations. This is not so. As seen in the introduction, the ideal MHD equations are invariant under Galilean transformations (more precisely, under the magnetic limit of the Lorenz transformations [75]). If

(u', b', π') are the physical quantities in a reference frame defined by a vector $V \in \mathbb{R}^3$, then they are expressed as a function of the original quantities (u, b, π) by

$$u'(t,x) = u(t, x - Vt) + V, \quad b'(t,x) = b(t, x - Vt), \quad \pi'(t,x) = \pi(t, x - Vt).$$

If (u, b, π) solves the ideal MHD equations, then so does (u', b', π'). This observation can be extended to a generalized Galilean invariance of MHD. For any given smooth function of time $f : \mathbb{R}_t \longrightarrow \mathbb{R}^3$, let F be a primitive of f and set

$$u'(t,x) = u(t, x - F(t)) + f(t), \quad b'(t,x) = b(t, x - F(t)), \quad \pi'(t,x) = \pi(t, x - F(t)) - f'(t) \cdot x. \tag{3.6}$$

Then the quantities (u, b, π) and (u', b', π') equivalently solve the ideal MHD equations. In particular, the magnetic field equation

$$\partial_t b' + (u' \cdot \nabla) b' - (b' \cdot \nabla) u' = 0$$

holds *without any added gradient term* in the righthand side. Similarly, the Elsässer system is also left unchanged by these transformations, provided that α' and β' are defined according to (3.6) and the pressures by $\pi'_k(t,x) = \pi_k(t, x - F(t)) - f'(t) \cdot x$ for $k = 1, 2$. Because both pressures π_k obey the same transformation law, $Q = \pi_2 - \pi_1$ is, just as the magnetic field for MHD, a (generalized) Galilean invariant of the Elsässer system.

The question remains open as to whether it is possible to obtain any solution of the Elsässer system as the image of a MHD solution by the operation of a certain group (which cannot be the Galilean group). Consequently, the precise nature of the "added" solutions in the Elsässer system remains, to us, somewhat mysterious, and is the object of our continued investigation.[4]

3.2.1 Weak Solutions

In this paragraph, we define the appropriate notions of weak solutions of systems (3.1) and (3.2). We start by the usual MHD system (3.1) before doing so for the Elsässer system.

Definition 3.7. Let $T > 0$ and $u_0, b_0 \in L^\infty$ be two divergence-free functions. A couple $(u, b) \in L^2_{\text{loc}}([0, T[; L^\infty)$ is said to be a *bounded weak solution* of (3.1), related to the initial data (u_0, b_0) if there exists a (MHD) pressure $\pi \in \mathcal{D}'([0, T[\times \mathbb{R}^d)$ such that

(i) the momentum equation is satisfied in the sense of distributions:

$$\partial_t u + \operatorname{div}(u \otimes u - b \otimes b) + \nabla \pi = \delta_0(t) \otimes u_0(x) \qquad \text{in } \mathcal{D}'([0, T[\times \mathbb{R}^d)\,;$$

we refer to Definition 2.11 for the meaning of the tensor product $\delta_0 \otimes u_0$;

(ii) the magnetic field equation is satisfied in the sense of distributions:

$$\partial_t b + \operatorname{div}(u \otimes b - b \otimes u) = \delta_0(t) \otimes b_0(x) \qquad \text{in } \mathcal{D}'([0, T[\times \mathbb{R}^d),$$

(iii) the divergence-free condition $\operatorname{div}(u) = 0$ holds in $\mathcal{D}'(]0, T[\times \mathbb{R}^d)$.

Definition 3.8. Let $T > 0$ and $\alpha_0, \beta_0 \in L^\infty$ be two divergence-free functions. A couple $(\alpha, \beta) \in L^2_{\text{loc}}([0, T[; L^\infty)$ is said to be a *bounded weak solution* of (3.2) related to the initial data (α_0, β_0) if there exist two pressure functions $\pi_1, \pi_2 \in \mathcal{D}'([0, T[\times \mathbb{R}^d)$ such that

(i) the equation for α is satisfied in the sense of distributions:

$$\partial_t \alpha + \operatorname{div}(\beta \otimes \alpha) + \nabla \pi_1 = \delta_0(t) \otimes \alpha_0(x) \qquad \text{in } \mathcal{D}'([0, T[\times \mathbb{R}^d),$$

and likewise for β;

[4] We are indebted to Christoph Charles lengthy discussions on the issues that concern us in this Remark and the one before. The unexpected depth of the topic and our own ignorance prevents us from expanding further.

(ii) both divergence-free conditions $\mathrm{div}(\alpha) = \mathrm{div}(\beta) = 0$ hold in $\mathcal{D}'(]0, T[\times\mathbb{R}^d)$.

Remark 3.9. As above with the Euler system, we have the same ambiguity concerning initial data. If (u, b) is as in Definition 3.7, then there is no need for u_0 and $u(0)$ to be the same, even if the solution lies in $C^1_T(L^\infty)$. Likewise, there is no reason for u to be continuous with respect to time. However, the magnetic field has a much more pleasant behavior: it automatically lies in $W^{1,1}_T(W^{-1,\infty})$, and we have $b_0 = b(0)$.

As for the Elsässer variables (α, β) in Definition 3.8, they share the same problems the Euler solutions may have: both initial values $\alpha(0)$ and $\beta(0)$ may differ from the initial data α_0, β_0, and both α and β may be discontinuous with respect to time.

Remark 3.10. As we have noted in [29] (Section 4.1), to define weak solutions, it is only necessary, strictly speaking, that $u \otimes u - b \otimes b$ and $u \otimes b - b \otimes u$ be in $L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$. Interestingly, this is absolutely equivalent to $\alpha \otimes \beta$ being locally integrable. But we will not need such a level of generality.

3.2.2 Equivalence of the Systems

The following statement is a precise sharp result for bounded solutions of the ideal MHD system (3.1) and (3.2).

Theorem 3.11. *Let $T > 0$, $\alpha_0, \beta_0 \in L^\infty$ and $(\alpha, \beta) \in C^0\big([0, T[; L^\infty\big)$. Define $(u, b) = \frac{1}{2}(\alpha + \beta, \alpha - \beta)$, and (u_0, b_0) accordingly. The assertions below are true.*

1. *Assume that (u, b) is a weak solution of* (3.1) *that is related to the initial data (u_0, b_0), in the sense of Definition 3.7. Then (α, β) solves the Elsässer system* (3.2) *with initial data (α_0, β_0) and $\pi_1 = \pi_2 = \pi$.*

2. *Assume that (α, β) is a weak solution of* (3.2) *for the initial data (α_0, β_0). Then the following statements are equivalent:*

   (i) the functions (u, b) solve the "classical" MHD system (3.1) *with initial data $(u_0, b(0))$, and the difference $b(0) - b_0$ is a constant function;*

   (ii) we have, for all times $t \in [0, T[$, $b(t) - b(0) \in \mathcal{S}'_h$.

   In addition, if one of these equivalent conditions is fulfilled, then the pressure gradients $\nabla\pi_1$ and $\nabla\pi_2$ are equal.

Proof. Because the proof of this theorem bears strong similarities with that of Theorem 2.18, we only give an outline of the arguments.

If (u, b) is a weak solution of (3.1), then adding and subtracting the weak forms of the momentum and magnetic field equations makes it clear that (α, β) solves the Elsässer system with initial data (α_0, β_0) with $\pi_1 = \pi_2$.

On the other hand, assume that (α, β) is a solution of the Elsässer system (3.2), according to Definition 3.8, and with the initial data (α_0, β_0). As in the proof of Theorem 2.18, extend the solution $(\alpha, \beta, \pi_1, \pi_2)$ to $] - \infty, T[\times\mathbb{R}^d$ and continue to note $(\alpha, \beta, \pi_1, \pi_2)$ the extension. By subtracting both equations for α and β, we see that the magnetic field satisfies

$$\partial_t b + \mathrm{div}(u \otimes b - b \otimes u) = \frac{1}{2}\nabla(\pi_2 - \pi_1) + \delta_0(t) \otimes b_0(x).$$

As with $C^0_T(L^\infty)$ solutions of the Euler system, we apply the same mollification sequence $\big(K_\epsilon\big)_{\epsilon>0}$ in order to deal with low regularity in the time variable. We have, by noting $(u_\epsilon, b_\epsilon) = K_\epsilon * (u, b)$ the convolution with respect to time,

$$\partial_t b_\epsilon + K_\epsilon * \mathrm{div}(u \otimes b - b \otimes u) = \frac{1}{2} K_\epsilon * \nabla(\pi_2 - \pi_1) + K_\epsilon(t) b_0(x) \qquad \text{in }] - \infty, T - \epsilon[\times\mathbb{R}^d.$$

By taking the divergence of this equation, we see that $K_\epsilon * \Delta(\pi_2 - \pi_1) = 0$, hence the functions $Q_\epsilon(t) = K_\epsilon * (\pi_2 - \pi_1)(t) \in \mathbb{R}[X]$ are harmonic polynomials. As in the proof of Theorem 2.18, we see that $\nabla Q_\epsilon = K_\epsilon * \nabla Q$ for some distribution $\nabla Q \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$. We must find a condition under which we have $\nabla Q = 0$ in $]0, T[\times \mathbb{R}^d$.

We now notice that the quadratic term in the magnetic field equation lies in $\mathcal{S}'_h$, as shown by the first Bernstein inequality (Lemma 1.7). For all times $t \in]0, T[$,

$$\|\chi(\lambda D) \operatorname{div}(u \otimes b - b \otimes u)\|_{L^\infty} = O\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda \to +\infty.$$

Therefore, by arguing with b exactly as with u at the end of the proof of Theorem 2.18, we get the equivalence of points *(i)* and *(ii)*. □

As with the Euler equations, well-posedness of a projected version of the ideal MHD system has been studied in the critical Besov space $B^1_{\infty,1}$ (see the work of Miao and Yuan [83] summarized in Theorem 3.3 in the discussion above). In the ideal MHD equations (3.1), the pressure only appears in the momentum equation. This means that, in order to apply the Leray projection we must require the velocity field u to satisfy a special condition. The resulting system is

$$\begin{cases} \partial_t u + \mathbb{P} \operatorname{div}(u \otimes u - b \otimes b) = 0 \\ \partial_t b + \operatorname{div}(u \otimes b - b \otimes u) = 0, \end{cases} \tag{3.7}$$

The projected system (3.7) is trivially equivalent to a projected system for the Elsässer variables, namely

$$\begin{cases} \partial_t \alpha + \mathbb{P} \operatorname{div}(\beta \otimes \alpha) = 0 \\ \partial_t \beta + \mathbb{P} \operatorname{div}(\alpha \otimes \beta) = 0. \end{cases} \tag{3.8}$$

However, solutions $(\alpha, \beta) \in C^0_T(L^\infty)$ of the Elsässer system, as in Definition 3.8, solve these projected equations, with appropriate initial data, if and only if

$$\big(\alpha(0), \beta(0)\big) = (\alpha_0, \beta_0) \qquad \text{and} \qquad \big(\alpha(t) - \alpha(0), \beta(t) - \beta(0)\big) \in \mathcal{S}'_h \text{ for } t \in [0, T[,$$

so that Theorem 3.11 remains a necessary step to recast the Elsässer system (3.1) into the classical MHD one. Relations between these different systems are summarized in the following diagram.

$$\begin{tikzcd}
\text{Els} \arrow[rr, "\alpha, \beta"] \arrow[d, "b"] & & \mathbb{P}(\text{Els}) \arrow[ll, bend left] \arrow[d, shift right] \\
\text{MHD} \arrow[u, bend left] \arrow[rr, "u"] & & \mathbb{P}(\text{MHD}) \arrow[ll, bend left] \arrow[u, shift right]
\end{tikzcd}$$

In this diagram, Els stands for the Elsässer system (3.2) while MHD stands for the classical MHD system (3.1). The names $\mathbb{P}$(Els) and $\mathbb{P}$(MHD) refer to, respectively, the projected equations (3.8) and (3.7). The arrows are labeled with the conditions required to pass from a system to another, bare arrows need no conditions.

3.3 Local Well-Posedness of Ideal MHD

In this Section, we prove well-posedness for the ideal MHD system in critical Besov spaces. This result has first been obtained by Miao and Yuan [83], but the proof we provide is much simpler, in addition of clarifying the uniqueness issue of bounded solutions.

Theorem 3.12. *Let $p, r \in [1, +\infty]$ and $s \in \mathbb{R}$ such that the Besov space $B^s_{p,r}$ is supercritical: $B^s_{p,r} \subset W^{1,\infty}$ or in other words*

$$s > 1 + \frac{d}{p} \qquad \text{or} \qquad s = 1 + \frac{d}{p} \text{ and } r = 1. \tag{3.9}$$

Consider divergence-free initial data $u_0, b_0 \in B^s_{p,r}$. There is a time $T > 0$ such that the system (3.1) has a unique solution $(u, b) \in C^0([0, T[; B^s_{p,r})$ if $r < +\infty$, and[5] in $C^0_w([0, T[; B^s_{p,r})$ if $r = +\infty$, associated to the initial data (u_0, b_0) which satisfies, if $p = +\infty$,

$$u(t) - u(0) \in \mathcal{S}'_h.$$

In addition, if $r < +\infty$, we also have $u, b \in C^1([0, T[; B^{s-1}_{p,r})$ and $\partial_t u, \partial_t b \in C^0_w([0, T[; B^{s-1}_{p,r})$ when $r = +\infty$. Finally, the time T can be chosen so that

$$T \geq \frac{C}{\|u_0, b_0\|_{B^s_{p,r}}},$$

for some constant $C = C(d, s, p, r)$.

3.3.1 *A Priori* Estimates

Consider (s, p, r) as in Theorem 3.12. In this paragraph, we focus on finding *a priori* estimates in the Besov space $B^s_{p,r}$ for smooth solutions of the projected ideal MHD equations written in Elsässer variables, as in (3.8) above:

$$\begin{cases} \partial_t \alpha + \mathbb{P} \operatorname{div}(\beta \otimes \alpha) = 0 \\ \partial_t \beta + \mathbb{P} \operatorname{div}(\alpha \otimes \beta) = 0. \end{cases} \tag{3.10}$$

Proposition 3.13. *Let $(s, p, r) \in \mathbb{R} \times [1, +\infty] \times [1, +\infty]$ be as in Theorem 3.12. Let (α, β) be regular solutions of system* (3.10)*, related to smooth initial data (α_0, β_0), with α_0 and β_0 being divergence-free. Then there exist two constants $C_1, C_2 > 0$, which depend on the dimension d and (s, p, r), as well as a time $T^* > 0$, which depends on the quantities above and on $\big\|(\alpha_0, \beta_0)\big\|_{B^s_{p,r}}$, such that*

$$\sup_{t \in [0, T^*]} \big\|(\alpha(t), \beta(t))\big\|_{B^s_{p,r}} \leq C_1 \big\|(\alpha_0, \beta_0)\big\|_{B^s_{p,r}}; \tag{3.11}$$

moreover, we have the inequality

$$T^* \geq \frac{C_2}{\big\|(\alpha_0, \beta_0)\big\|_{B^s_{p,r}}}.$$

The general method of the proof is very close to the theory of transport equations in Besov spaces. In fact, we can see (3.10) as a system of (coupled) transport equations by introducing commutator terms:

$$\begin{cases} \partial_t \alpha + (\beta \cdot \nabla) \alpha = \big[\beta \cdot \nabla, \mathbb{P}\big] \alpha \\ \partial_t \beta + (\alpha \cdot \nabla) \beta = \big[\alpha \cdot \nabla, \mathbb{P}\big] \beta. \end{cases}$$

As for transport equations, to get $B^s_{p,r}$ estimates we will apply the Littlewood-Paley blocks Δ_j to the equations. By adding new commutators, we see that $\Delta_j(\alpha, \beta)$ also solves (forced) transport equations: for instance, on α we have

$$\Big(\partial_t + \beta \cdot \nabla\Big) \Delta_j \alpha = \Delta_j \big[\beta \cdot \nabla, \mathbb{P}\big] \alpha + \big[\beta \cdot \nabla, \Delta_j\big] \alpha. \tag{3.12}$$

[5]The symbol $C^0_w([0, T[; X)$ refers to measurable functions $f : [0, T[\longrightarrow X$ with values in a Banach space with a predual Y which are continuous for the weak topology: for all $\phi \in Y$, the bracket $t \mapsto \langle f(t), \phi \rangle_{X \times Y}$ is continuous on $[0, T[$.

Therefore, precise estimations of all the commutator terms will be crucial in the proof. This is the scope of the next three Lemmata. The first one is designed to handle the first commutator in the equation above, and is inescapable when dealing with transport equations while in the last two we focus on the last commutator $\big[\beta\cdot\nabla,\mathbb{P}\big]\alpha$ in (3.12) above: Lemma 3.17 shows that if $\alpha,\beta\in B^s_{p,r}$ then it is bounded in the same space.

Lemma 3.14 (Lemma 2.100 and Remark 2.101 in [7]). *Assume that $v\in B^s_{p,r}$, with (s,p,r) satisfying the Lipschitz condition* (3.9). *Then*

$$\forall f\in B^s_{p,r},\qquad 2^{js}\big\|\big[v\cdot\nabla,\Delta_j\big]f\big\|_{L^p}\lesssim c_j\Big(\|\nabla v\|_{L^\infty}\|f\|_{B^s_{p,r}}+\|\nabla v\|_{B^{s-1}_{p,r}}\|\nabla f\|_{L^\infty}\Big),$$

for some sequence $\big(c_j\big)_{j\geq-1}$ in the unit ball of ℓ^r.

Lemma 3.15 (Lemma 2.99 in [7]). *Let κ be a smooth function on $\mathbb{R}^d$, which is homogeneous of degree m away from a neighborhood of* 0. *Then, for a vector field v such that $\nabla v\in L^\infty$, one has:*

$$\forall f\in B^s_{p,r},\qquad \big\|\big[\mathcal{T}_v,\kappa(D)\big]f\big\|_{B^{s-m+1}_{p,r}}\lesssim\|\nabla v\|_{L^\infty}\|f\|_{\dot B^s_{p,r}}.$$

Remark 3.16. It is in fact no problem to apply Lemma 3.15 to the Leray projector $\kappa(D)=\mathbb{P}$ even if its symbol isn't smooth at $\xi=0$, as the low frequency values of $\kappa(\xi)$ are not involved in the commutator:

$$\big[\mathcal{T}_v,\kappa(D)\big]f=\sum_{m\geq-1}\Big(S_{m-1}v.\Delta_m\kappa(D)f-\kappa(D)\big(S_{m-1}v.\Delta_m f\big)\Big).$$

For $m=-1$, the first term in the sum vanishes so that $\kappa(D)$ is only applied to blocks $\Delta_m f$ with $m\geq0$, and next, the product $S_{m-1}v.\Delta_m f$ is always spectrally supported away from $\xi=0$. This has nothing surprising, as the paraproduct was constructed in this way (see Section 1.5). Therefore, we may serenely write

$$\forall f\in B^s_{p,r},\qquad \big\|\big[\mathcal{T}_v,\mathbb{P}\big]f\big\|_{B^{s+1}_{p,r}}\lesssim\|\nabla v\|_{L^\infty}\|f\|_{B^s_{p,r}}.$$

Lemma 3.17. *Let $(s,p,r)\in\mathbb{R}\times[1,+\infty]\times[1,+\infty]$ satisfy conditions* (3.9). *For all divergence-free vector fields $\alpha,\beta\in\dot B^s_{p,r}$, we have*

$$\big\|\big[\beta\cdot\nabla,\mathbb{P}\big]\alpha\big\|_{B^s_{p,r}}\lesssim\|\alpha\|_{B^s_{p,r}}\|\beta\|_{B^s_{p,r}}$$

Remark 3.18. The alert reader will have noticed that the inequality of Lemma 3.17 is not optimal: it only involves a product of $B^s_{p,r}$ norms, instead of a mix of low and high order norms as Lemmata 3.15 and 3.14 above. In fact, there are subtle issues linked to the endpoint exponent $p=+\infty$, so we concentrate for the moment on a simpler estimate which leads to well-posedness. Later, in Section 3.4, we will work on improving Lemma 3.17 in order to obtain continuity criteria.

Proof of Lemma 3.17. The estimate is based on the Bony decomposition (see Section 1.5 in Chapter 1) for the two products appearing in the commutator, which yields

$$\big[\beta\cdot\nabla,\mathbb{P}\big]\alpha=\big[\mathcal{T}_{\beta_k}\partial_k,\mathbb{P}\big]\alpha+\mathcal{T}_{\partial_k\mathbb{P}\alpha}(\beta_k)-\mathbb{P}\mathcal{T}_{\partial_k\alpha}(\beta_k)+\mathcal{R}\big(\beta_k,\partial_k\mathbb{P}\alpha\big)-\mathbb{P}\mathcal{R}\big(\beta_k,\partial_k\alpha\big),$$

where an implicit summation is made over the repeated index $k=1,...,d$. To begin with, Lemma 3.15 (along with Remark 3.16) allows us to handle the first summand:

$$\big\|\big[\mathcal{T}_{\beta_k},\mathbb{P}\big]\partial_k\alpha\big\|_{B^s_{p,r}}\lesssim\|\nabla\alpha\|_{B^{s-1}_{p,r}}\|\nabla\beta\|_{L^\infty}\lesssim\|\alpha\|_{B^s_{p,r}}\|\nabla\beta\|_{L^\infty}.$$

Next, we look at the remaining two paraproduct terms $\mathcal{T}_{\partial_k\mathbb{P}\alpha}(\beta_k)$ and $\mathbb{P}\mathcal{T}_{\partial_k\alpha}(\beta_k)$. We start by showing that we indeed have $\mathbb{P}\partial_k\alpha=\partial_k\alpha$. This is a consequence of the fact that the derivative

$\partial_k\alpha$ is an element of $\mathcal{S}'_h$, and so is given by its homogeneous Littlewood-Paley decomposition, on which the Leray projector acts regularly:

$$\mathbb{P}\partial_k\alpha = \sum_{m\in\mathbb{Z}} \mathbb{P}\dot{\Delta}_m\partial_k\alpha = \sum_{m\in\mathbb{Z}} \dot{\Delta}_m\partial_k\alpha = \partial_k\alpha.$$

We may therefore write the paraproduct $\mathcal{T}_{\partial_k\mathbb{P}\alpha}(\beta_k)$ as $\mathcal{T}_{\partial_k\alpha}(\beta_k)$ and use Proposition 1.20 to obtain

$$\|\mathcal{T}_{\partial_k\mathbb{P}\alpha}(\beta_k)\|_{B^s_{p,r}} = \|\mathcal{T}_{\partial_k\alpha}(\beta_k)\|_{B^s_{p,r}} \lesssim \|\nabla\alpha\|_{L^\infty}\|\beta\|_{B^s_{p,r}}.$$

For the second paraproduct, we note that $\mathcal{T}_{\partial_k\alpha}(\beta_k)$ is spectrally supported away from $\xi = 0$, since all the terms involved in the sum

$$\mathcal{T}_{\partial_k\alpha}(\beta_k) = \sum_{m\geq -1} S_{m-1}\partial_k\alpha.\Delta_m\beta_k$$

are spectrally supported in annuli. Therefore, the Leray projection operator acts continuously on $\mathcal{T}_{\partial_k\alpha}(\beta_k)$ in the Besov topology $B^s_{p,r}$ and we have

$$\|\mathbb{P}\mathcal{T}_{\partial_k\alpha}(\beta_k)\|_{B^s_{p,r}} \lesssim \|\nabla\alpha\|_{L^\infty}\|\beta\|_{B^s_{p,r}}.$$

Finally, we turn to the two remainder terms. As we have noted just above, since $\mathbb{P}\partial_k\alpha = \partial_k\alpha$, there is no problem in bounding the first remainder: by Proposition 1.21, we have

$$\|\mathcal{R}(\beta_k,\mathbb{P}\partial_k\alpha)\|_{B^s_{p,r}} = \|\mathcal{R}(\beta_k,\partial_k\alpha)\|_{B^s_{p,r}} \lesssim \|\nabla\alpha\|_{B^0_{\infty,\infty}}\|\beta\|_{B^s_{p,r}} \lesssim \|\nabla\alpha\|_{L^\infty}\|\beta\|_{B^s_{p,r}}.$$

For the second remainder, we need to pay attention to the way the Leray projector operates on low frequencies. While $\mathbb{P}\Delta_{-1}$ is not necessarily bounded on L^p if $p = 1$ or $p = +\infty$, Proposition 2.15 shows that the operator $\mathbb{P}\partial_k\Delta_{-1}$ is bounded on every L^p for $p \in [1, +\infty]$, that is including the endpoint cases $p = 1$ or $p = +\infty$. We deduce that

$$\begin{aligned}
\big\|\mathbb{P}\partial_k\mathcal{R}(\beta_k,\alpha)\big\|_{B^s_{p,r}} &\lesssim \|\Delta_{-1}\mathbb{P}\partial_k\mathcal{R}(\beta_k,\alpha)\|_{L^p} + \left(\sum_{j\geq 0} 2^{jsr}\|\Delta_j\mathbb{P}\mathcal{R}(\beta_k,\partial_k\alpha)\|^r_{L^p}\right)^{1/r}\\
&\lesssim \|\mathcal{R}(\beta,\alpha)\|_{L^p} + \|\mathcal{R}(\beta_k,\partial_k\alpha)\|_{B^s_{p,r}}\\
&\lesssim \|\beta\|_{B^s_{p,r}}\|\alpha\|_{B^0_{\infty,\infty}} + \|\beta\|_{B^s_{p,r}}\|\nabla\alpha\|_{B^0_{\infty,\infty}}.
\end{aligned}$$

with the usual modification if $r = +\infty$. The lemma is thus proved. □

Proof of Proposition 3.13. We are finally ready to prove Proposition 3.13. We fix an index $j \geq -1$ and apply the dyadic block Δ_j to the equations of system (3.10) in order to obtain equation (3.12) and the corresponding equation on β, namely

$$\begin{cases}\Big(\partial_t + \beta\cdot\nabla\Big)\Delta_j\alpha = \big[\beta\cdot\nabla,\, \Delta_j\big]\alpha + \Delta_j\big[\beta\cdot\nabla,\, \mathbb{P}\big]\alpha\\ \Big(\partial_t + \alpha\cdot\nabla\Big)\Delta_j\beta = \big[\alpha\cdot\nabla,\, \Delta_j\big]\beta + \Delta_j\big[\alpha\cdot\nabla,\, \mathbb{P}\big]\beta.\end{cases}$$

Now, Lemma 3.14 gives the following bounds: there exists a sequence $\big(c_j(t)\big)_{j\geq -1}$ in the unit ball of ℓ^r such that we have the following inequality:

$$2^{sj}\Big(\big\|\big[\beta\cdot\nabla,\,\Delta_j\big]\alpha\big\|_{L^p} + \big\|\big[\alpha\cdot\nabla,\,\Delta_j\big]\beta\big\|_{L^p}\Big) \lesssim c_j(t)\left(\|\alpha\|_{B^s_{p,r}}\|\beta\|_{B^s_{p,r}}\right).$$

On the other hand, Lemma 3.17 yields a similar inequality: there exists a (possibly different) sequence $\big(c_j(t)\big)_j$ in the unit ball of ℓ^r such that

$$\big\|\Delta_j\big[\beta\cdot\nabla,\mathbb{P}\big]\alpha\big\|_{L^p} + \big\|\Delta_j\big[\alpha\cdot\nabla,\mathbb{P}\big]\beta\big\|_{L^p} \lesssim c_j(t)\|\alpha\|_{B^s_{p,r}}\|\beta\|_{B^s_{p,r}}.$$

Therefore, basic L^p estimates for transport equations with divergence-free vector fields give an integral inequality for the dyadic blocks:

$$2^{js}\big\|\Delta_j\big(\alpha(t),\beta(t)\big)\big\|_{L^p} \lesssim 2^{js}\big\|\Delta_j\big(\alpha_0,\beta_0\big)\big\|_{L^p} + \int_0^t c_j(\tau)\big\|\big(\alpha(\tau),\beta(\tau)\big)\big\|^2_{B^s_{p,r}}\,\mathrm{d}\tau\,.$$

Taking the ℓ^r norm, we find, for all times $t>0$, the inequality

$$\big\|\big(\alpha(t),\beta(t)\big)\big\|_{L^\infty_t(B^s_{p,r})} \lesssim \|(\alpha_0,\beta_0)\|_{B^s_{p,r}} + \left\|\int_0^t c_j(\tau)\big\|\big(\alpha(\tau),\beta(\tau)\big)\big\|^2_{B^s_{p,r}}\,\mathrm{d}\tau\right\|_{\ell^r}\,,$$

and using the Minkowski inequality (see Proposition 1.3 in [7]) to slip the ℓ^r norm inside the integral, we finally deduce the following bound:

$$c\,\big\|\big(\alpha(t),\beta(t)\big)\big\|_{B^s_{p,r}} \leq \|(\alpha_0,\beta_0)\|_{B^s_{p,r}} + \int_0^t \big\|\big(\alpha(\tau),\beta(\tau)\big)\big\|^2_{B^s_{p,r}}\,\mathrm{d}\tau\,. \tag{3.13}$$

for some constant $c>0$ depending only on (d,s,p,r).

From this inequality, it is easy to obtain an estimate like (3.11) in some interval $[0,T^*]$. For this, set $E(t)=\big\|\big(\alpha(t),\beta(t)\big)\big\|_{B^s_{p,r}}$ and define the time T^* by

$$T^* = \sup\left\{T>0\;\Big|\quad \int_0^T E(t)^2\,\mathrm{d}t < E(0)\right\}\,.$$

Then, from (3.13) we get estimate (3.11) in $[0,T^*]$:

$$\forall\, t\in[0,T^*]\,,\qquad\qquad E(t)\;\leq\;\frac{2}{c}\,E(0)\,.$$

In turn, this latter inequality provides also a lower bound for T^*. Indeed, for all $t\leq T^*$ we have

$$\int_0^t E(\tau)\mathrm{d}\tau\;\leq\;\left(\frac{2}{c}\,E(0)\right)^2\,t\,,$$

so that, by definition of T^*, we must have $T^*\geq c^2/\big(4\,E(0)\big)$.

This ends the proof of the proposition. □

We end this section with a few remarks on the regularity of the pressure term in the specific setting of incompressible fluids. Since we work with the projected system (3.10), the pressure gradient $\nabla\pi_1=\nabla\pi_2$ is given by

$$\nabla\pi_1 = (\mathrm{Id}-\mathbb{P})\,\mathrm{div}(\beta\otimes\alpha).$$

By using the tame estimates (1.12) for the Banach algebra $B^s_{p,r}$ (recall that $s\geq1$), we may obtain (with a bit of additional work in the endpoint cases $p=1$ or $p=+\infty$) that, if $T\leq T^*$ where T^* is given by Proposition 3.13,

$$\|\nabla\pi_1\|_{L^\infty_T(B^{s-1}_{p,r})} \lesssim \|\alpha_0,\beta_0\|^2_{B^s_{p,r}}.$$

However, by taking advantage of the algebraic structure induced by both divergence conditions $\mathrm{div}(\alpha)=\mathrm{div}(\beta)=0$, we may improve this bound.

Corollary 3.19. *Consider (d,s,p,r), (α_0,β_0) and (α,β) exactly as in Proposition 3.13. Let T^* be as defined in that same proposition. Then, we have the following estimates for the (equal) pressure terms: for all $T\in[0,T^*]$,*

$$\|\nabla\pi_1\|_{L^\infty_T(B^s_{p,r})} = \|\nabla\pi_2\|_{L^\infty_T(B^s_{p,r})} \lesssim \big\|(\alpha_0,\beta_0)\big\|^2_{B^s_{p,r}}.$$

Proof. The main argument of the proof is that, owing to the fact that both α and β are divergence free, there is a number of simplifications when taking derivatives of the product $\beta \otimes \alpha$, namely $\partial_j\partial_k(\beta_k\alpha_j) = \partial_j\beta_k\partial_k\alpha_j$. Therefore, by writing the Bony decomposition for the product, we obtain

$$\begin{aligned}\nabla\pi_1 &= \nabla(-\Delta)^{-1}\partial_j\partial_k(\beta_k\alpha_j) = \nabla(-\Delta)^{-1}\partial_j\beta_k\partial_k\alpha_j \\ &= \nabla(-\Delta)^{-1}\mathcal{T}_{\partial_j\beta_k}(\partial_k\alpha_j) + \nabla(-\Delta)^{-1}\mathcal{T}_{\partial_k\alpha_j}(\partial_j\beta_k) + \nabla(-\Delta)^{-1}\partial_j\partial_k\mathcal{R}(\beta_k,\alpha_j).\end{aligned}$$

Recall that the paraproducts are, by construction, spectrally supported away from $\xi = 0$ (see the proof of Lemma 3.17), so that the homogeneous Fourier multipliers such as $\nabla(-\Delta)^{-1}$ acts continuously on them in non-homogeneous Besov spaces (by Lemma 1.5). This means that

$$\|\nabla(-\Delta)^{-1}\mathcal{T}_{\partial_j\beta_k}(\partial_k\alpha_j)\|_{B^s_{p,r}} + \|\nabla(-\Delta)^{-1}\mathcal{T}_{\partial_k\alpha_j}(\partial_j\beta_k)\|_{B^s_{p,r}} \lesssim \|\nabla\alpha\|_{L^\infty}\|\beta\|_{B^s_{p,r}} + \|\nabla\beta\|_{L^\infty}\|\alpha\|_{B^s_{p,r}}.$$

The remainder term is a bit more delicate to bound. Firstly, we must be careful when dealing with low frequencies, as Fourier multipliers may not be well behaved there: we will use Proposition 2.15 in order to obtain boundedness in L^p (for the whole range $p \in [1,+\infty]$) of the order 1 operator $\Delta_{-1}\nabla(-\Delta)^{-1}\partial_j\partial_k$. On the other hand, to deal with high frequencies, we note that $s \geq 1$ so that derivatives $\nabla\mathcal{R}(\alpha_j,\beta_k)$ of the remainder have $2s-1 \geq s$ regularity by Proposition 1.21. All in all, by use of Lemma 1.5, we have, with the usual modification if $r = +\infty$,

$$\begin{aligned}\|\nabla(-\Delta)^{-1}\partial_j\partial_k\mathcal{R}(\alpha_j,\beta_k)\|_{B^s_{p,r}} &\leq \|\Delta_{-1}\nabla(-\Delta)^{-1}\partial_j\partial_k\mathcal{R}(\alpha_j,\beta_k)\|_{L^p} \\ &\qquad + \left(\sum_{j\geq 0} 2^{jsr}\|\Delta_j\nabla(-\Delta)^{-1}\partial_j\partial_k\mathcal{R}(\alpha_j,\beta_k)\|^r_{L^p}\right)^{1/r} \\ &\lesssim \|\mathcal{R}(\alpha,\beta)\|_{L^p} + \|\nabla\mathcal{R}(\alpha,\beta)\|_{B^s_{p,r}} \lesssim \|\alpha\|_{B^s_{p,r}}\|\beta\|_{B^s_{p,r}}.\end{aligned}$$

Resorting to the estimates of Proposition 3.13 ends the proof or Corollary 3.19. □

3.3.2 Existence and Uniqueness of Solutions

Let us sum up what we have obtained so far, and close the proof to Theorem 3.12.

Uniqueness. Uniqueness of solutions to system (3.10) in the class $B^s_{p,r}$, under conditions (3.9) was shown in [83]. This was achieved by using (somehow implicitly) Elsässer variables. This is enough, in view of the equivalence established in Theorem 3.11 above, so, for the sake of brevity, we will not give details here, although we point out that in the next Chapter, we use an energy method to show uniqueness of finite energy solutions with Besov-Lipschitz regularity (see Theorem 4.7).

Existence. Also for this, we invoke the result of [83], in particular for what concerns the construction of smooth approximate solutions to (3.10) and their convergence to a true solution of that problem. Therefore, we limit ourselves to show only *a priori* bounds for regular solutions (u,b) in Besov norms: those estimates are given by Proposition 3.13. In the next Chapter, we will use different *a priori* estimates and provide a full proof of existence by leans of an iterative scheme (see Subsection 4.3.2).

Besov regularity of the time derivatives and of the pressure term. The Besov regularity $B^{s-1}_{p,r}$ of the time derivatives $(\partial_t u, \partial_t b)$ follows by noting that $\partial_t\alpha = -\mathbb{P}\,\mathrm{div}(\beta\otimes\alpha)$, and likewise for $\partial_t\beta$, then using the Besov regularity for (α,β) and product rules in Besov spaces. The $B^s_{p,r}$ regularity on the pressure terms $\nabla\pi$ is described in Corollary 3.19.

Time regularity properties. Time regularity properties stated in Theorem 3.12 are direct consequences of the solution's Besov regularity and standard results on transport equations in Besov spaces (see e.g. Theorem 3.19 in [7]). This having been established, one can bootstrap the time regularity of the time derivatives $(\partial_t u, \partial_t b)$ and of the pressure term, in the same way as explained above.

3.4 Continuation Criteria

In this Section, we give a number of continuation criteria in the spirit of Beale-Kato-Majda [8] (BKM below) for the solutions constructed in Theorem 3.12. We will start by a classical result involving the integral of $\|\nabla u, \nabla b\|_{L^\infty}$ before broadening our discussion in two ways.

First, we will see that the endpoint case $p = +\infty$ involves unexpected difficulties, which we will discuss in Subsection 3.4.2. Next, we will try to find criteria which involve the norm of a single vector quantity, such as $\nabla^2 u$ or $u \pm b$.

3.4.1 A BKM-type result

In this section, we seek to prove a continuation criterion for the solutions of (3.10) in $B^s_{p,r}$ under the assumption that $1 < p < +\infty$. We will of course, as before, assume that the Lipschitz condition (3.9) holds on the indices (s, p, r).

Proposition 3.20. *Assume that* $1 < p < +\infty$ *and let* $(\alpha_0, \beta_0) \in B^s_{p,r}$*, with* $\mathrm{div}(\alpha_0) = \mathrm{div}(\beta_0) = 0$*. Given a time* $T > 0$*, let* (α, β) *be a solution of* (3.10) *on* $[0, T[$*, related to that initial datum and belonging to* $L^\infty_t(B^s_{p,r})$ *for any* $0 \le t < T$*. Assume that*

$$\int_0^T \Big\{ \|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} \Big\} \, \mathrm{d}t < +\infty. \tag{3.14}$$

Then (α, β) *can be continued beyond* T *into a solution of* (3.10) *with the same regularity.*

Remark 3.21. This theorem implies that the lifespan of solutions in $B^s_{p,r}$ (with (s, p, r) satisfying the Lipschitz condition, *i.e.* condition (3.9) above) does not depend on the values of (s, p, r) for $1 < p < +\infty$, thanks to the embeddings $B^s_{p,r} \subset B^{1-d/p}_{\infty,r} \subset B^1_{\infty,1} \subset W^{1,\infty}$. In other words, the regularity of the initial datum is propagated until the solution explodes.

Proof. By standard arguments, it is enough to show that a solution (α, β) remains bounded in $L^\infty_T(B^s_{p,r})$ as long as T satisfies (3.14). This is mainly a matter of rewriting the inequalities of the previous proof in a more precise way.

Let (α, β) be a solution of (3.10) in $[0, T[$ related to the initial data (α_0, β_0) as in Proposition 3.20. We already know from Lemma 3.14 that

$$\begin{aligned} 2^{js}\Big(\big\| [\beta \cdot \nabla, \, \Delta_j]\alpha \big\|_{L^p} + \big\| [\alpha \cdot \nabla, \, \Delta_j]\beta \big\|_{L^p} \Big) \\ \lesssim c_j(t) \Big(\|\nabla\alpha\|_{L^\infty} \|\beta\|_{B^s_{p,r}} + \|\alpha\|_{B^s_{p,r}} \|\nabla\beta\|_{L^\infty} \Big) \end{aligned} \tag{3.15}$$

for some sequence $\big(c_j(t)\big)_j$ belonging to the unit sphere of ℓ^r. The other commutator $\big[\beta \cdot \nabla, \mathbb{P}\big]\alpha$ requires a bit more work, as we have to improve the inequality of Lemma 3.17. Recall that, by writing as in the proof of the Lemma, the Bony decomposition for $\big[\beta \cdot \nabla, \mathbb{P}\big]\alpha$, we get again

$$\begin{aligned} \big[\beta \cdot \nabla, \mathbb{P}\big]\alpha &= \big[\mathcal{T}_{\beta_k}\partial_k, \mathbb{P}\big]\alpha + \mathcal{T}_{\partial_k \mathbb{P}\alpha}(\beta_k) - \mathbb{P}\mathcal{T}_{\partial_k\alpha}(\beta_k) + \mathcal{R}\big(\beta_k, \partial_k\mathbb{P}\alpha\big) - \mathbb{P}\mathcal{R}\big(\beta_k, \partial_k\alpha\big) \\ &:= F_{\beta,\alpha} - \mathbb{P}\mathcal{R}\big(\beta_k, \partial_k\alpha\big). \end{aligned}$$

and likewise for $\big[\alpha\cdot\nabla,\mathbb{P}\big]\beta$. By proceeding exactly as in the proof of Lemma 3.17, we find that both $F_{\alpha,\beta}$ and $F_{\beta,\alpha}$ can be bounded by

$$2^{js}\Big(\,\|\Delta_j F_{\alpha,\beta}\|_{L^p}+\|\Delta_j F_{\beta,\alpha}\|_{L^p}\,\Big)\lesssim c_j(t)\,\Big(\|\nabla\alpha\|_{L^\infty}\|\beta\|_{B^s_{p,r}}+\|\alpha\|_{B^s_{p,r}}\|\nabla\beta\|_{L^\infty}\Big)\,. \tag{3.16}$$

The remainders $\mathbb{P}\mathcal{R}(\beta_k,\partial_k\alpha)$ and $\mathbb{P}\mathcal{R}(\alpha_k,\partial_k\beta)$ are the last terms that are left for us to bound: Proposition 1.21 gives

$$\|\mathcal{R}(\beta_k,\partial_k\alpha)\|_{B^s_{p,r}}+\|\mathcal{R}(\alpha_k,\partial_k\beta)\|_{B^s_{p,r}}\lesssim\|\nabla\alpha\|_{B^0_{\infty,\infty}}\|\beta\|_{B^s_{p,r}}+\|\nabla\beta\|_{B^0_{\infty,\infty}}\|\alpha\|_{B^s_{p,r}},$$

and because we work with Lebesgue exponents $1<p<+\infty$, we know that the Leray projection operator is bounded on L^p by Theorem 1.23, and therefore is also bounded on the Besov space $B^s_{p,r}$. We deduce that

$$\|\mathbb{P}\mathcal{R}(\beta_k,\partial_k\alpha)\|_{B^s_{p,r}}+\|\mathbb{P}\mathcal{R}(\alpha_k,\partial_k\beta)\|_{B^s_{p,r}}\lesssim\|\nabla\alpha\|_{B^0_{\infty,\infty}}\|\beta\|_{B^s_{p,r}}+\|\nabla\beta\|_{B^0_{\infty,\infty}}\|\alpha\|_{B^s_{p,r}}. \tag{3.17}$$

Putting estimates (3.15), (3.16) and (3.17) together, we are led to the following bound:

$$\big\|\big(\alpha(t),\beta(t)\big)\big\|_{B^s_{p,r}}\lesssim\big\|(\alpha_0,\beta_0)\big\|_{B^s_{p,r}}+\int_0^t\Big(\|\nabla\alpha\|_{L^\infty}+\|\nabla\beta\|_{L^\infty}\Big)\,\big\|(\alpha,\beta)\big\|_{B^s_{p,r}}\mathrm{d}\tau$$

The desired property is a simple application of Grönwall's lemma:

$$\sup_{t\in[0,T[}\big\|\big(\alpha(t),\beta(t)\big)\big\|_{B^s_{p,r}}\lesssim 1\,.$$

As already said at the beginning of the proof, by standard arguments and uniqueness of solutions, this latter inequality implies the result. □

3.4.2 The Endpoint Case $p=+\infty$

We now focus on the endpoint case $p=+\infty$. To our surprise, we have been unable to prove Proposition 3.20 in that case without additional "low frequency" assumptions.

The difficulty in adapting the computations above lies in finding bounds for the commutator $\big[\beta\cdot\nabla,\mathbb{P}\big]\alpha$ (and the corresponding one obtained by exchanging α and β), as Lemma 3.17 only provides an inequality that involves the *non-homogeneous* norms of α and β instead of the norms of their derivatives. This is a stronger information because

$$\forall f\in B^s_{p,r},\qquad\|\nabla f\|_{B^{s-1}_{p,r}}\lesssim\|f\|_{B^s_{p,r}},$$

the reverse inequality being false. More precisely, inequality (3.17) in the proof of Proposition 3.20 does not hold if $p=+\infty$.

To provide continuation criteria in the case $p=+\infty$, we will add *low-frequency* assumptions, such as additional integrability on the initial data. This is the object of the next three statements, which represent previously unpublished work.

Proposition 3.22. *Consider (s,r) such that $B^s_{\infty,1}\subset W^{1,\infty}$, that is $s>1$ or $s=1$ and $r=1$. Let $(u_0,b_0)\in B^s_{\infty,r}$ be a set of divergence free initial data and let $(u,b)\in C^0([0,T[;B^s_{\infty,r})$, with the usual change to C^0_w if $r=+\infty$, be the unique solution of* (3.10) *related to the initial data (u_0,b_0) as given by Theorem 3.12. Then (u,b) may be extended beyond time T into a solution with the same regularity provided that*

$$\int_0^T\Big\{\|\Delta_{-1}(u,b)\|_{L^\infty}+\|(\nabla u,\nabla b)\|_{L^\infty}\Big\}\,\mathrm{d}t<+\infty.$$

Proof. By reviewing the proof of Proposition 3.20, we see that the only troublesome term are the remainder $\mathcal{R}(\beta_k, \partial_k\alpha)$ and $\mathcal{R}(\alpha_k, \partial_k\beta)$ for which we had used the $L^p \longrightarrow L^p$ boundedness of the Leray projection operator. However, when $p = +\infty$, this is no longer possible, and we only have the bound of Lemma 3.17, which does not imply the derivatives $\nabla\alpha$ or $\nabla\beta$.

However, thanks to the continuity of the operator $\Delta_{-1}\mathbb{P}\partial_k : L^\infty \longrightarrow L^\infty$ (see Proposition 2.15), we may write a more precise estimate: by use of Proposition 1.21, of the second Bernstein inequality (Lemma 1.7) and of the embeddings $B^0_{\infty,1} \subset L^\infty \subset B^0_{\infty,\infty}$ we have

$$\begin{aligned}
\|\mathcal{R}(\beta_k, \partial_k\alpha)\|_{B^s_{\infty,r}} &\lesssim \|\Delta_{-1}\mathbb{P}\partial_k\mathcal{R}(\beta_k, \alpha)\|_{L^\infty} + \|(\mathrm{Id} - \Delta_{-1})\mathbb{P}\mathcal{R}(\beta_k, \partial_k\alpha)\|_{B^s_{\infty,r}} \\
&\lesssim \|\mathcal{R}(\beta, \alpha)\|_{L^\infty} + \|\mathcal{R}(\beta_k, \partial_k\alpha)\|_{B^s_{p,r}} \\
&\lesssim \|\beta\|_{B^1_{\infty,1}}\|\alpha\|_{B^0_{\infty,\infty}} + \|\beta\|_{B^s_{\infty,r}}\|\nabla\alpha\|_{B^0_{\infty,\infty}} \\
&\lesssim \|\beta\|_{B^s_{\infty,r}}\Big(\|\Delta_{-1}\alpha\|_{L^\infty} + \|\nabla\alpha\|_{L^\infty}\Big).
\end{aligned}$$

By combining this (and the corresponding inequality for $\mathcal{R}(\alpha_k, \partial_k\beta)$) with the other inequalities in the proof of Proposition 3.20, we have a differential inequality for $\|(\alpha, \beta)\|_{B^s_{\infty,r}}$ involving the integrand in the statement of Proposition 3.22, namely:

$$\|(\alpha, \beta)\|_{L^\infty_t(B^s_{\infty,r})} \lesssim \|(\alpha, \beta)\|_{B^s_{\infty,r}} + \int_0^t \|(\alpha, \beta)\|_{B^s_{\infty,r}}\Big(\|\Delta_{-1}(\alpha, \beta)\|_{L^\infty} + \|(\nabla\alpha, \nabla\beta)\|_{L^\infty}\Big)\mathrm{d}\tau.$$

A direct application of Grönwall's lemma ends the proof. □

Corollary 3.23. *With the assumptions of Proposition 3.22, assume in addition that $u_0, b_0 \in L^p$ for some finite $p < +\infty$. Then the solution (u, b) may be extended beyond time T if and only if*

$$\int_0^T \Big\{\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty}\Big\}\,\mathrm{d}t < +\infty. \tag{3.18}$$

Proof. First, we can assume that $p > 1$, as straightforward interpolation gives the inclusion $(\alpha_0, \beta_0) \in L^1 \cap B^s_{\infty,r} \subset L^1 \cap W^{1,\infty} \subset L^q$ for any $q > 1$. According to Proposition 3.22 and with the help of the first Bernstein inequality (Lemma 1.7), it is enough to show that the L^p norm of the solution (α, β) remains bounded under condition (3.18). For this, we consider α and β as solutions of transport equations:

$$\Big(\partial_t + \beta\cdot\nabla\Big)\alpha = (\mathrm{Id} - \mathbb{P})(\beta\cdot\nabla)\alpha,$$

and likewise for β. Because the Leray projection operator is $L^p \longrightarrow L^p$ bounded by Theorem 1.23, we have:

$$\|\alpha\|_{L^p} \leq \|\alpha_0\|_{L^p} + C\int_0^t \|(\beta\cdot\nabla)\alpha\|_{L^p}\mathrm{d}\tau,$$

with an equivalent inequality for β. Using Grönwall's lemma gives

$$\|(\alpha, \beta)\|_{L^\infty_T(L^p)} \leq \|(\alpha_0, \beta_0)\|_{L^p}\exp\left(C\int_0^T \|(\nabla\alpha, \nabla\beta)\|_{L^\infty}\,\mathrm{d}t\right),$$

and ends the proof. □

In the proof of Proposition 3.22, the whole difficulty is to define the low frequency block $\Delta_{-1}\mathbb{P}\partial_k\mathcal{R}(\beta_k, \alpha)$, the derivative ∂_k must be factored *out* of the remainder, against the Leray projector whereas keeping the derivative *inside* the remainder yields, by the $L^\infty \longrightarrow \dot{B}^0_{\infty,\infty}$ boundedness of $\mathbb{P}$,

$$\|\mathbb{P}\mathcal{R}(\beta_k, \partial_k\alpha)\|_{\dot{B}^0_{\infty,\infty}} \lesssim \|\mathcal{R}(\beta_k, \partial_k\alpha)\|_{\dot{B}^0_{\infty,\infty}} \lesssim \|\beta\|_{B^1_{\infty,1}}\|\nabla\alpha\|_{B^0_{\infty,\infty}}, \tag{3.19}$$

which is insufficient for our purposes. On the other hand, factoring the derivative outside gives a bound for $\Delta_{-1}\mathbb{P}\partial_k\mathcal{R}(\beta_k,\alpha)$ that is far stronger than needed:

$$\|\Delta_{-1}\mathbb{P}\partial_k\mathcal{R}(\beta_k,\alpha)\|_{\dot{B}^{-1}_{\infty,\infty}} \lesssim \|\mathcal{R}(\beta,\alpha)\|_{L^\infty} \lesssim \|\beta\|_{B^1_{\infty,1}}\|\alpha\|_{B^0_{\infty,\infty}}, \tag{3.20}$$

as the operator $\dot{S}_0 = \sum_{m\leq -1}\dot{\Delta}_m$ maps the space $\dot{B}^{-\epsilon}_{\infty,\infty}$ continuously to L^∞ for any $\epsilon > 0$. Although using fractional derivatives is not an option for finding a middle ground between those two inequalities, as only the order one divergence operator provides the *algebraic* simplification $\partial_k(\beta_k\alpha) = \beta_k\partial_k\alpha$, logarithmic interpolation allows us to relax a bit the criterion of Proposition 3.22 as to the time regularity of the low frequency blocks.

Proposition 3.24. *With the notations of Proposition 3.22, the solution (u,b) may be continued beyond time T in a solution with the same regularity provided that*

$$\int_0^T \|(\nabla u,\nabla b)\|_{L^\infty}\left\{1+\log\left(1+\frac{\|\Delta_{-1}(u,b)\|_{L^\infty}}{\|(\nabla u,\nabla b)\|_{L^\infty}}\right)\right\}\,\mathrm{d}t < +\infty.$$

Proof. Consider a fixed $\eta\in]0,1[$. We note $R = \Delta_{-1}\mathbb{P}\partial_k\mathcal{R}(\beta_k,\alpha)$ the block which we must bound and let $N\geq 0$ whose value will be chosen later. By separating low and high frequencies, we have, since $R\in\mathcal{S}'_h$ is the sum of its Littlewood-Paley decomposition,

$$\|R\|_{L^\infty} \leq \sum_{-N}^{0}\|\dot{\Delta}_jR\|_{L^\infty} + \sum_{-\infty}^{-N-1}\|\dot{\Delta}_jR\|_{L^\infty} \leq (N+1)\|R\|_{\dot{B}^0_{\infty,\infty}} + \sum_{-\infty}^{-N-1}2^{j\eta}c_j\|R\|_{\dot{B}^{-\eta}_{\infty,1}},$$

where $(c_j)_{j\in\mathbb{Z}}$ is a nonnegative sequence in the unit ball of $\ell^1(\mathbb{Z})$. By virtue of the inequalities (3.19) and (3.20) above, we may bound R by

$$\|R\|_{L^\infty} \lesssim \|\beta\|_{B^1_{\infty,1}}\left\{(N+1)\|\nabla\alpha\|_{L^\infty} + 2^{-N\eta}\|\alpha\|_{B^0_{\infty,\infty}}\right\}.$$

We now fix the value of N so that both terms in the brackets are comparable, that is

$$N = \frac{1}{\eta}\log_2\left(\frac{\|\alpha\|_{B^0_{\infty,\infty}}}{\|\nabla\alpha\|_{L^\infty}}\right)$$

so as to obtain

$$\|R\|_{L^\infty} \lesssim \frac{1}{\eta}\|\beta\|_{B^1_{\infty,1}}\|\nabla\alpha\|_{L^\infty}\left\{1+\log\left(\frac{\|\alpha\|_{B^0_{\infty,\infty}}}{\|\nabla\alpha\|_{L^\infty}}\right)\right\}.$$

Now, we can bound the norm of α in the logarithm as we did in the proof of Proposition 3.22 by $\|\alpha\|_{B^0_{\infty,\infty}} \lesssim \|\Delta_{-1}\alpha\|_{L^\infty} + \|\nabla\alpha\|_{L^\infty}$ and use the fact that log is an increasing function to get

$$\|R\|_{L^\infty} \lesssim \frac{1}{\eta}\|\beta\|_{B^1_{\infty,1}}\|\nabla\alpha\|_{L^\infty}\left\{1+\log\left(1+\frac{\|\Delta_{-1}\alpha\|_{L^\infty}}{\|\nabla\alpha\|_{L^\infty}}\right)\right\}.$$

After doing the exact same manipulations for the other remainder, we may end our argument by combining this with the other inequalities in the proof of Proposition 3.22 and using Grönwall's lemma. □

3.4.3 A Velocity-Based Criterion

Although the ideal MHD system presents a very interesting and symmetric structure in Elsässer variables (3.2), the system in its original form (3.1) also has clear advantages. Amongst them is that the equation for the magnetic field is both *linear* and *local* (it only involves derivatives of the unknowns whereas the pressure terms in the Elsässer system are non-local). This is an indication that blow-up of a regular solution can be detected by a condition on the velocity only.

Theorem 3.25. *Let $(u_0, b_0) \in L^2(\mathbb{R}^d) \cap B^1_{\infty,1}(\mathbb{R}^d)$ be a set of divergence-free initial data. Consider $T > 0$ such that the ideal MHD system, supplemented with those initial data, has a unique solution (u, b) in the space $C^0\big([0, T[\,; L^2(\mathbb{R}^d) \cap B^1_{\infty,1}(\mathbb{R}^d)\big)$ according to Theorem 3.12. Then this solution may be continued beyond time T provided that*

$$\int_0^T \left\|\nabla^2 u(t)\right\|_{L^\infty} \,\mathrm{d}t < +\infty. \tag{3.21}$$

Proof. As far as continuation results go (keep in mind Corollary 3.23), we already know that the solution may be prolonged beyond time T if we have

$$\int_0^T \Big\{ \|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} \Big\} \,\mathrm{d}t < +\infty. \tag{3.22}$$

Therefore, we only have to show that this integral is finite under condition (3.21). We start by recalling that, by a simple energy method, we have

$$\sup_{t \in [0,T[} \Big(\|u(t)\|_{L^2} + \|b(t)\|_{L^2} \Big) \lesssim \|u_0\|_{L^2} + \|b_0\|_{L^2}. \tag{3.23}$$

By using this bound together with the Bernstein inequalities (Lemma 1.7), we can estimate

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\leq \|\Delta_{-1} \nabla u\|_{L^\infty} + \sum_{m \geq 0} \|\Delta_m \nabla u\|_{L^\infty} \lesssim \|u\|_{L^2} + \sum_{m \geq 0} 2^{-m} \|\Delta_m \nabla^2 u\|_{L^\infty} \\ &\lesssim \big\|(u_0, b_0)\big\|_{L^2} + \|\nabla^2 u\|_{L^\infty}. \end{aligned} \tag{3.24}$$

Thus, under condition (3.21) we deduce that $\|\nabla u\|_{L^1_T(L^\infty)} \lesssim T \,\big\|(u_0, b_0)\big\|_{L^2} + \|\nabla^2 u\|_{L^1_T(L^\infty)} < +\infty$. It remains us to show that $\|\nabla b\|_{L^1_T(L^\infty)}$ is also finite. This will be a consequence of the fact that b solves a linear transport equation, which we may differentiate to obtain estimates on the first derivative ∇b. Precisely, for $j = 1, ..., d$, we have

$$\partial_t \partial_j b + (u \cdot \nabla) \partial_j b = -(\partial_j u \cdot \nabla) b + (\partial_j b \cdot \nabla) u + (b \cdot \nabla) \partial_j u.$$

A basic L^∞-estimate immediately gives, for all $0 \leq t < T$, the bound

$$\|\nabla b(t)\|_{L^\infty} \leq \|\nabla b_0\|_{L^\infty} + \int_0^t \Big\{ \|\nabla u\|_{L^\infty} \|\nabla b\|_{L^\infty} + \|b\|_{L^\infty} \|\nabla^2 u\|_{L^\infty} \Big\} \mathrm{d}\tau \tag{3.25}$$

The term $\|\nabla u\|_{L^\infty}$ has already been estimated in (3.24). Arguing similarly, we can find an upper bound for $\|b\|_{L^\infty}$ which involve only the quantities we have at our disposal: by separating low and high frequencies, we have

$$\begin{aligned} \|b\|_{L^\infty} &\leq \|\Delta_{-1} b\|_{L^\infty} + \sum_{m \geq 0} \|\Delta_m b\|_{L^\infty} \lesssim \|b\|_{L^2} + \sum_{m \geq 0} 2^{-m} \|\Delta_m \nabla b\|_{L^\infty} \\ &\lesssim \big\|(u_0, b_0)\big\|_{L^2} + \|\nabla b\|_{L^\infty}. \end{aligned}$$

Plugging (3.24) and the previous bound into (3.25), we obtain an integral inequality which is linear with respect to $\|\nabla b\|_{L^\infty}$: for all $0 \leq t < T$, we have

$$\begin{aligned} \sup_{\tau \in [0,t]} \|\nabla b(\tau)\|_{L^\infty} \lesssim \|\nabla b_0\|_{L^\infty} &+ \big\|(u_0, b_0)\big\|_{L^2} \int_0^t \|\nabla^2 u\|_{L^\infty} \mathrm{d}\tau \\ &+ \int_0^t \|\nabla b\|_{L^\infty} \Big(\big\|(u_0, b_0)\big\|_{L^2} + \|\nabla^2 u\|_{L^\infty} \Big) \mathrm{d}\tau. \end{aligned}$$

By using Grönwall's lemma, we deduce that $\|\nabla b\|_{L^\infty}$ must be bounded as long as condition (3.21) is fulfilled. So the integral (3.22) must therefore also be finite while (3.21) holds. $\square$

We conclude this part with a remark concerning a continuation criterion in terms of the magnetic field only.

Remark 3.26. The evolution of the velocity field is dictated by the momentum equation, which is quadratic with respect to u. This implies that it is not possible, in general, to find the kind of linear estimates that would yield a continuation criterion dispensing of any condition on the velocity field. However, in the special case of space dimension $d = 2$, the vorticity equation is linear in the derivatives of u: we have

$$\partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j \,,$$

where $j = \partial_1 b_2 - \partial_2 b_1$ is the electrical current crossing the plane of the fluid. Therefore, in that case it is possible to bound u, or ω, given good enough bounds on the magnetic field. This leads to a continuation criterion based only on b: we have $T < T^*$ as long as

$$\int_0^T \|j\|^2_{B^1_{\infty,1}} \,\mathrm{d}t < +\infty.$$

3.4.4 A Elsässer-Based Criterion

Similarly to the magnetic field, the equations for the Elsässer variables are also *linear*, although they are coupled. This implies that we are able to find continuation criteria based on either $\alpha = u + b$ or $\beta = u - b$.

Theorem 3.27. *Let $(u_0, b_0) \in L^2(\mathbb{R}^d) \cap B^1_{\infty,1}(\mathbb{R}^d)$ be a set of divergence-free initial data. Consider $T > 0$ such that the ideal MHD system, supplemented with those initial data, has a unique solution (u, b) in the space $C^0\big([0, T[\,; L^2(\mathbb{R}^d) \cap B^1_{\infty,1}(\mathbb{R}^d)\big)$. Then, if we denote $\omega = \mathrm{curl}\,(u)$ and $j = \mathrm{curl}\,(b)$, one has*

$$\int_0^T \big\| \, \omega + j \, \big\|_{B^0_{\infty,1}} \,\mathrm{d}t < +\infty \qquad \Longleftrightarrow \qquad \int_0^T \big\| \, \omega - j \, \big\|_{B^0_{\infty,1}} \,\mathrm{d}t < +\infty \,.$$

In addition, in the case where those integrals are finite, the solution (u, b) may be continued beyond T into a solution belonging to the same regularity class.

Proof. We already know from Proposition 3.20 that the solution may be continued beyond the time T if and only if

$$\int_0^T \Big\{ \|\nabla \alpha\|_{L^\infty} + \|\nabla \beta\|_{L^\infty} \Big\} \,\mathrm{d}t < +\infty. \tag{3.26}$$

Throughout this proof, we assume that

$$\int_0^T \big\| \, \omega + j \, \big\|_{B^0_{\infty,1}} \,\mathrm{d}t < +\infty. \tag{3.27}$$

We are going to show that this condition is enough to ensure that we also have

$$\int_0^T \big\| \, \omega - j \, \big\|_{B^0_{\infty,1}} \,\mathrm{d}t < +\infty \,, \tag{3.28}$$

and that the finiteness of both integrals implies (3.26). The exact same argument will apply also when considering the quantity $\omega - j$, whence the claimed equivalence. We start the proof by remarking that, splitting into low and high frequencies as done in (3.24), using the Biot-Savart law

$$f_k \; = \; (-\Delta)^{-1} \sum_{j=1}^d \partial_j \big[\mathrm{curl}\,(f)\big]_{jk} \,,$$

and Lemma 1.5, it is easy to show that, for any divergence-free vector field f, one has

$$\|\nabla f\|_{L^\infty} \leq \|f\|_{B^1_{\infty,1}} \lesssim \|f\|_{L^2} + \|\mathrm{curl}\,(f)\|_{B^0_{\infty,1}} . \tag{3.29}$$

In the above equations, we have denoted $\mathrm{curl}\,(f)$ the matrix such that $\big[\mathrm{curl}\,(f)\big]_{jk} = \partial_j f_k - \partial_k f_j$, with the usual identification $\mathrm{curl}\,(f) = \nabla \times f$ in dimension $d = 3$, and $\mathrm{curl}\,(f) = \partial_1 f_2 - \partial_2 f_1$ in dimension $d = 2$. Thus, after noticing that

$$\|\alpha\|_{L^2} \leq \big\|(u, b)\big\|_{L^2} \leq \big\|(u_0, b_0)\big\|_{L^2}$$

in view of the energy inequality (3.23), we get

$$\|\nabla(u + b)\|_{L^\infty} \leq \|u + b\|_{B^1_{\infty,1}} \lesssim \big\|(u_0, b_0)\big\|_{L^2} + \|\omega + j\|_{B^0_{\infty,1}} . \tag{3.30}$$

So, it remains us to show that the integral in (3.28) remains finite under condition (3.27). For this, we note that the solution (α, β) satisfies the *a priori* estimates of Proposition 3.13. In particular, we must also have (see inequality (3.13))

$$\|\beta\|_{L^\infty_T(B^1_{\infty,1})} \lesssim \|\beta_0\|_{B^1_{\infty,1}} + \int_0^T \|\alpha\|_{B^1_{\infty,1}} \|\beta\|_{B^1_{\infty,1}}\, \mathrm{d}t.$$

At this point, we estimate the $B^1_{\infty,1}$ norm of $\alpha = u + b$ by using (3.30), and we finally infer an integral inequality which is linear with respect to $\|\beta\|_{B^1_{\infty,1}}$:

$$\|\beta\|_{L^\infty_T(B^1_{\infty,1})} \lesssim \|\beta_0\|_{B^1_{\infty,1}} + \int_0^T \Big(\|\alpha_0\|_{L^2} + \|\,\omega + j\,\|_{B^0_{\infty,1}}\Big) \|\beta\|_{B^1_{\infty,1}}\, \mathrm{d}t\, .$$

Thus, we may use Grönwall's lemma to end the proof. □

Remark 3.28. Because of the presence of the pressure terms in the equations and of the fact that we work with solutions that may reach critical regularity $s \geq 1 + d/p$, we are unable to obtain a continuation criterion depending only on the L^∞ norm of $\nabla(u \pm b)$.

However, in a subcritical regularity framework $B^s_{\infty,r}$ with $s > 1$, we believe that the same method of [8] applies to give a continuation criterion in terms of the finiteness of the norm $\|\mathrm{curl}\,(u \pm b)\|_{L^1_T(L^\infty)}$. We do not treat the extension of our result in this direction here.

3.5 Improved Bounds for the Lifespan

In this section, we present an improved lower bound for the lifespan of solutions for a plane fluid $d = 2$. As we have explained above, in the case of $d = 2$, the Euler equations are globally well-posed, so, for continuity considerations, it is natural to assume that the lifespan of ideal MHD solutions would increase to $T = +\infty$ in the regime of low magnetic fields. Our lower bound takes this into account.

Theorem 3.29. *Let $(u_0, b_0) \in L^2(\mathbb{R}^2)$ be a set of divergence-free initial data such that $u_0 \in B^2_{\infty,1}(\mathbb{R}^2)$ and $b_0 \in B^1_{\infty,1}(\mathbb{R}^2)$. Then, the lifespan $T > 0$ of the corresponding solution $(u, b) \in C\big([0, T[\,; L^2(\mathbb{R}^2) \cap B^1_{\infty,1}(\mathbb{R}^2)\big)$ of the 2-D ideal MHD system (3.1), given by Theorem 3.12, enjoys the following lower bound:*

$$T \geq \frac{C}{\|u_0\|_{L^2 \cap B^2_{\infty,1}}} \log\left\{1 + C\, \log\left[1 + C\, \log\left(1 + C\frac{\|u_0\|_{L^2 \cap B^2_{\infty,1}}}{\|b_0\|_{B^1_{\infty,1}}}\right)\right]\right\},$$

where $C > 0$ is a constant independent of the initial data.

Remark 3.30. Our result is stated in two dimensions of space. This is crucial, as the proof relies on the existence of global solutions to the Euler system. However, this is the only point in our argument that is specific to $d = 2$, and our proof may be adapted to all dimensions $d \geq 2$ to show the following fact: if we denote by $T_E > 0$ the lifespan of the unique $B^2_{\infty,1}$ solution v of the Euler problem with initial datum $v_{|t=0} = u_0$, and if we set $\|b_0\|_{B^1_{\infty,1}} = \varepsilon$, then the lifespan T_ε of the solution (u, b) to the ideal MHD system (3.1) satisfies

$$\varliminf_{\epsilon\to 0^+} T_\varepsilon \geq T_E$$

For this reason, we believe general method of the proof should be adapted in different frameworks: formally, the lifespan of solutions to a family of PDEs is expected, under good conditions, to have some sort of continuity with respect to the parameters involved in the PDEs (here, the size of the magnetic field).

Remark 3.31. The statement of Theorem 3.29 has different regularity and integrability assumptions for the velocity and magnetic fields. This is somewhat unexpected because we work with with a system of (non-local) transport equations, so all unknowns should have the same regularity because of the coupling. However, the method of our proof implies that we only work with $B^1_{\infty,1}$ solutions (α, β) of the ideal MHD equations. The $B^2_{\infty,1}$ assumption on u_0 is only used to construct a regular solution v of the Euler equations to which we will compare (α, β).

3.5.1 Well-Posedness of the 2D Euler Equations

To begin with, we solve the incompressible Euler equations with initial datum u_0. More precisely, let $v : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the unique global solution of the initial value problem

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \nabla p = 0 \\ \mathrm{div}(v) = 0 \\ v_{|t=0} = u_0 \end{cases} \tag{3.31}$$

which lies in the class $C^0\big(\mathbb{R}_+; L^2(\mathbb{R}^2) \cap B^2_{\infty,1}(\mathbb{R}^2)\big)$. We refer *e.g.* to Chapter 7 of [7] for details. In the rest of the proof, we will need order two $B^2_{\infty,1}$ estimates on the velocity field, which are contained in the following lemma. Even though the bound therein is well-known, we were not able to find a precise reference for it. Therefore, we also provide a full proof.

Lemma 3.32. *Set $V_0 = \|u_0\|_{L^2\cap B^2_{\infty,1}}$. Then, for all $T > 0$, we have the following inequality:*

$$\sup_{t\in[0,T]} \|v(t)\|_{L^2\cap B^2_{\infty,1}} \leq C\, V_0 \exp\left(C\,T\,V_0\, e^{C\,T\,V_0}\right) ,$$

for some numerical constant $C > 0$, independent of u_0.

Proof. Since the following energy conservation holds true, namely

$$\forall\, t \geq 0\,, \qquad \|v(t)\|_{L^2} \leq \|v_0\|_{L^2}\,, \tag{3.32}$$

we only have to bound the Besov norm of v. For this, we resort to the vorticity form of the Euler equations. Define the vorticity $\Omega = \partial_1 v_2 - \partial_2 v_1$ of the flow v, which can be recovered from Ω by the 2-D Biot-Savart law $v = -\nabla^\perp(-\Delta)^{-1}\Omega$. Then, Ω solves the pure transport equation

$$\partial_t \Omega + v \cdot \nabla\Omega = 0\,, \qquad \text{with} \qquad \Omega_{|t=0} = \Omega_0 := \partial_1 v_{0,2} - \partial_2 v_{0,1}. \tag{3.33}$$

Transport equations such as (3.33) above provide uniform bounds for solutions in Lebesgue L^p spaces, for instance $\|\Omega\|_{L^p} = \|\Omega_0\|_{L^p}$, whereas estimates for higher order quantities, such as

$\|\Omega\|_{W^{s,p}}$, involve bounds that are exponential with respect to the transport field v. However, when seeking to estimate Ω in the space $B^0_{\infty,1}$, which is very similar to the Lebesgue space L^∞, one gets an estimate that is linear with respect to the transport field (this fact was discovered by Vishik [101], see also [64] and Theorem 3.18 in [7]). In our case, we have, for any $T > 0$, the bound

$$\|\Omega\|_{L^\infty_T(B^0_{\infty,1})} \lesssim \|\Omega_0\|_{B^0_{\infty,1}} \left(1 + \int_0^T \|\nabla v\|_{L^\infty} \,\mathrm{d}t\right). \tag{3.34}$$

To control the norm of the gradient ∇v appearing in this estimate, we resort to the inequality exhibited in (3.29): by combining the latter with (3.32), and then using Grönwall's lemma, from (3.34) we obtain

$$\|\Omega\|_{L^\infty_T(B^0_{\infty,1})} \lesssim \|\Omega_0\|_{B^0_{\infty,1}} \Big(1 + T\|u_0\|_{L^2}\Big) \exp\Big(cT\|\Omega_0\|_{B^0_{\infty,1}}\Big) .$$

So, by adding the energy $\|v\|_{L^2}$ to both sides of this inequality, we obtain a first order estimate of the solution v, namely

$$\begin{aligned} \|v\|_{L^\infty_T(L^2\cap B^1_{\infty,1})} &\lesssim \|u_0\|_{L^2} + \|\Omega_0\|_{B^0_{\infty,1}} \Big(1 + T\|u_0\|_{L^2}\Big) \exp\Big(cT\|\Omega_0\|_{B^0_{\infty,1}}\Big) \\ &\lesssim V_0(1 + TV_0)e^{cTV_0} \lesssim V_0 e^{cTV_0} . \end{aligned} \tag{3.35}$$

Next, we have to perform $B^1_{\infty,1}$ estimates for Ω. For this, it is enough to apply standard Besov estimates for transport equations to equation (3.33) (see Theorem 3.14 in [7]). In this way, we obtain

$$\|\Omega\|_{L^\infty_T(B^1_{\infty,1})} \lesssim \|\Omega_0\|_{B^1_{\infty,1}} \exp\left(C\int_0^T \|\nabla v\|_{B^0_{\infty,1}} \,\mathrm{d}t\right) \lesssim V_0 \exp\Big(cTV_0\, e^{cTV_0}\Big) ,$$

where we have make use of inequality (3.35) for passing from the first to the second inequality. Now, we complete the proof of the lemma by observing that $\|v\|_{B^2_{\infty,1}} \lesssim \|v\|_{L^2} + \|\Omega\|_{B^1_{\infty,1}}$. □

3.5.2 Proof of Theorem 3.29

The above estimates for solutions of the 2D Euler problem having been established, we come back to the ideal MHD system (3.1). Since the solution given by Theorem 3.12 is also a solution of the projected Elsässer system (3.10), we may work with (α, β) as a set of variables. Our strategy of proof is to compare the Elsässer system to the homogeneous Euler equations (3.31). In order to do so, we consider $v : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ the unique $C^0(B^2_{\infty,1})$ solution of the Euler equations associated to the initial datum u_0 as in Subsection 3.5.1 above and define the difference functions

$$\delta\alpha = \alpha - v = (u - v) + b \qquad \text{and} \qquad \delta\beta = \beta - v = (u - v) - b .$$

By using equations (3.10) and (3.31), we find that the couple $(\delta\alpha, \delta\beta)$ satisfies the (projected) system

$$\begin{cases} \partial_t(\delta\alpha) + \mathbb{P}(\beta\cdot\nabla)\delta\alpha + \mathbb{P}(\delta\beta\cdot\nabla)v = 0 \\ \partial_t(\delta\beta) + \mathbb{P}(\alpha\cdot\nabla)\delta\beta + \mathbb{P}(\delta\alpha\cdot\nabla)v = 0, \end{cases} \tag{3.36}$$

The equations in (3.36) can immediately be seen to be well-suited for our purposes. Firstly, the initial values $\delta\alpha_0$ and $\delta\beta_0$ of the difference functions depend only on the initial magnetic field:

$$\delta\alpha(0) = b_0 \qquad \text{and} \qquad \delta\beta(0) = -b_0.$$

In addition, estimating $\delta\alpha$ and $\delta\beta$ will immediately provide control for the solution (α, β) of the Elsässer system (3.10), thanks to explicit bounds for the regular solution v. With that in mind, we seek to estimate the $B^1_{\infty,1}$ norms of $(\delta\alpha,\delta\beta)$. To do this, we see from (3.36) that we will need $B^2_{\infty,1}$ bounds on v, which are given in Lemma 3.32 above.

Let $j\geq -1$. By introducing the commutators $[\beta\cdot\nabla,\mathbb{P}]\delta\alpha$ and $[\alpha\cdot\nabla,\mathbb{P}]\delta\beta$ and applying the dyadic block Δ_j to the first two equations in (3.36), we obtain

$$\begin{cases}\partial_t\Delta_j(\delta\alpha)+(\beta\cdot\nabla)\Delta_j(\delta\alpha)+\Delta_j\mathbb{P}(\delta\beta\cdot\nabla)v=\Delta_j\big[\beta\cdot\nabla,\mathbb{P}\big]\delta\alpha+\big[\beta\cdot\nabla,\Delta_j\big]\delta\alpha\\ \partial_t\Delta_j(\delta\beta)+(\alpha\cdot\nabla)\Delta_j(\delta\beta)+\Delta_j\mathbb{P}(\delta\alpha\cdot\nabla)v=\Delta_j\big[\alpha\cdot\nabla,\mathbb{P}\big]\delta\beta+\big[\alpha\cdot\nabla,\Delta_j\big]\delta\beta\,.\end{cases}$$

We now perform L^∞ estimates on that system. The commutators in the righthand side of the equations can be estimated by using Lemmata 3.14 and 3.17, so we only need to bound the products $\mathbb{P}(\delta\beta\cdot\nabla)v$ and $\mathbb{P}(\delta\alpha\cdot\nabla)v$. By separating low and high frequencies and using the $L^\infty\longrightarrow L^\infty$ boundedness of the operator $\Delta_{-1}\mathbb{P}\partial_k$ from Proposition 2.15, we obtain

$$\begin{aligned}\|\Delta_j\mathbb{P}(\delta\beta\cdot\nabla)v\|_{B^1_{\infty,1}}&\lesssim\|\Delta_{-1}\mathbb{P}\partial_k(\delta\beta_k v)\|_{L^\infty}+\|(\mathrm{Id}-\Delta_{-1})\mathbb{P}(\delta\beta\cdot\nabla)v\|_{B^1_{\infty,1}}\\&\lesssim\|\delta\beta_k v\|_{L^2}+\|(\delta\beta\cdot\nabla)v\|_{B^1_{\infty,1}}\lesssim\|u_0\|_{L^2}\|\delta\beta\|_{B^1_{\infty,1}}+\|(\delta\beta\cdot\nabla)v\|_{B^1_{\infty,1}}.\end{aligned}$$

We write the Bony decomposition for the product $(\delta\beta\cdot\nabla)v$ and use Propositions 1.20 and 1.21 so as to obtain

$$\begin{aligned}\|(\delta\beta\cdot\nabla)v\|_{B^1_{\infty,1}}&\lesssim\|\mathcal{T}_{\delta\beta_k}(\partial_k v)\|_{B^1_{\infty,1}}+\|\mathcal{T}_{\partial_k v}(\delta\beta_k)\|_{B^1_{\infty,1}}+\|\partial_k\mathcal{R}(\delta\beta_k,v)\|_{B^1_{\infty,1}}\\&\lesssim\|\delta\beta\|_{B^1_{\infty,1}}\|v\|_{B^2_{\infty,1}}\end{aligned}$$

Combining all the inequalities with the commutator estimates of Lemmata 3.14 and 3.17, we obtain a differential inequality on the dyadic blocks:

$$2^j\|\Delta_j(\delta\alpha,\delta\beta)\|_{L^\infty}\lesssim 2^j\|\Delta_j b_0\|_{L^\infty}+\int_0^t c_j(\tau)\|(\delta\alpha,\delta\beta)\|_{B^1_{\infty,1}}\Big(\|v\|_{B^2_{\infty,1}}+\|(\alpha,\beta)\|_{B^1_{\infty,1}}\Big)\mathrm{d}\tau,$$

where $\big(c_j(\tau)\big)_{j\geq -1}$ are sequences all belonging to the unit sphere of ℓ^1. By applying the Minkowski inequality to this last inequality, we obtain

$$\|(\delta\alpha,\delta\beta)\|_{B^1_{\infty,1}}\lesssim\|b_0\|_{B^1_{\infty,1}}+\int_0^t\|(\delta\alpha,\delta\beta)\|_{B^1_{\infty,1}}\Big(\|v\|_{B^2_{\infty,1}}+\|(\alpha,\beta)\|_{B^1_{\infty,1}}\Big)\mathrm{d}\tau\,.\tag{3.37}$$

With the previous estimate (3.37) at hand, we can conclude the proof of Theorem 3.29. In order to simplify the next computations, we define, for all $T>0$, the quantities

$$E(T)=\sup_{t\in[0,T]}\big\|(\delta\alpha(t),\delta\beta(t))\big\|_{B^1_{\infty,1}}\qquad\text{and}\qquad\phi(T)=\sup_{t\in[0,T]}\|v(t)\|_{B^2_{\infty,1}}\,.$$

For completing the proof, it remains us to estimate the $B^1_{\infty,1}$ norm of the solution (α,β) and find an inequality for $E(t)$, since $\phi(t)$ is finite at every time $t>0$. Now, we observe that $\alpha=\delta\alpha+v$ and $\beta=\delta\beta+v$, which implies

$$\big\|\big(\alpha(t),\beta(t)\big)\big\|_{B^1_{\infty,1}}\;\leq\;E(t)\;+\;\phi(t)\,.$$

Using this estimate in (3.37), we infer the integral inequality

$$E(t)\lesssim E_0+\int_0^t E(\tau)\Big(E(\tau)+\phi(\tau)\Big)\mathrm{d}\tau=E_0+\int_0^t E^2(\tau)\mathrm{d}\tau+\int_0^t E(\tau)\phi(\tau)\mathrm{d}\tau.$$

To get rid of the linear part in this inequality, namely the last summand in the right-hand side, we start by using Grönwall's lemma. Thus we get, for all $T>0$, the bound

$$E(T)\lesssim\left(E_0+\int_0^T E^2(t)\mathrm{d}t\right)e^{cT\phi(T)}\,,$$

where $c > 0$ is some irrelevant numerical constant. Next, in order to bound $E(T)$ on some time interval, we define the time $T^* > 0$ as

$$T^* := \sup \left\{ T > 0 \Big| \quad \int_0^T E^2(t) \mathrm{d}t \leq E_0 \right\}.$$

So, for all times $0 \leq T \leq T^*$, we must have the bound $E(T) \lesssim E_0 e^{cT\phi(T)}$. Therefore, by definition of T^*, we must have the inequality

$$e^{2cT^*\phi(T^*)}\, T^* \, \geq \, \frac{1}{E_0}\,.$$

This inequality alone proves that the time T^* on which the solution is known to satisfy uniform $B^1_{\infty,1}$ estimates is arbitrarily large if E_0 is made as small as necessary. However, to find more quantitative inequalities, we need to use the upper bound for $\phi(T)$ provided by Lemma 3.32: we find that, for all $T \leq T^*$, one has

$$\int_0^T E^2(t)\, \mathrm{d}t \lesssim \frac{E_0^2}{V_0} T V_0 \exp\left\{ CTV_0 \exp\left(CTV_0 e^{CTV_0} \right) \right\}.$$

Thus, by definition of T^*, for $T = T^*$ we must have

$$\frac{E_0^2}{V_0} T^* V_0 \exp\left\{ CT^* V_0 \exp\left(CT^* V_0 e^{CT^* V_0} \right) \right\} \, \geq \, E_0\,.$$

By using the inequality $x \leq e^x - 1$ in the previous estimate and applying the logarithm function three times, we prove the theorem.

Chapter 4

Vorticity Form of the Elsässer Variables

If people do not believe that mathematics is simple,
it is only because they do not realize how complicated life is.
John von Neumann[1]

4.1 Introduction

The purpose of this chapter is to explore the vorticity form of ideal MHD: equations which bear on the matrix variables

$$\omega_{ij} = [\operatorname{curl}(u)]_{ij} = \partial_j u_i - \partial_i u_j \qquad \text{and} \qquad j_{ij} = [\operatorname{curl}(b)]_{ij} = \partial_j b_i - \partial_i - b_j. \tag{4.1}$$

In three dimensions of space, the skew-symmetric matrix $\operatorname{curl}(u)$ is equivalently represented by vorticity vector $\omega = \nabla \times u$, and the matrix $\operatorname{curl}(b)$ by the electrical current $j = \nabla \times b$. In two dimensions of space, the matrices ω and j can be reduced to scalar quantities, which are the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ and electrical currents $j = \partial_1 b_2 - \partial_2 b_1$ that cross the plane of the fluid. We refer to page 36 in the Introduction for background on the meaning of plane MHD.

The advantages gathered from working in a vorticity formulation are numerous and well-known in incompressible hydrodynamics: for instance, it has been successfully used a number of times for proving major results concerning the Euler equations, such as the solutions of Yudovich (see [81], Chapter 8, for a detailed presentation of this celebrated result) or the bounded solutions of Serfati [97], [3]. We also mention the works of Vishik [101], Hmidi-Keraani [64] and Danchin-Fanelli [34] whose methods have a major influence in this dissertation, in particular concerning the use of critical Besov spaces and the pursuit of logarithmic lower bounds for the lifespan of solutions as in Theorem 3.29 above.

In this Chapter, we will focus on the algebraic properties of the equations solved by the curls of the Elsässer variables $X = \omega + j$ and $Y = \omega - j$ and show how most of the results of the previous Chapter can be attained in this way. Although we will not, strictly speaking, prove anything that has not been discussed above, we believe that at least two reasons advocate in favor of our attempt. Firstly, from a chronological point of view, Theorem 3.29 in the previous Chapter was first proved in our article [29] by using these vorticity techniques[2], which are closer to the [34] from which Theorem 3.29 is inspired. Secondly, on a mathematical perspective, we hope that a different algebraic form of our problem may provide insight regarding global well-posedness of

[1] This quote is attributed to von Neumann in B. Winckler, *Blagues mathématiques*. Ellipses, Paris, 2011.

[2] In fact, the material in this [29] has proved on a more general quasi-homogeneous MHD system, which we will derive as a limit of a fast-rotating problem in the next Chapter (see also Remark 4.2). The presence of a density perturbation function introduces a few technical difficulties. For the sake of conciseness and to pinpoint the harder issues, we have elected to only work with the simpler ideal MHD equations here and refer to [29] for the more general system.

plane MHD, which has been, and remains, one of the most challenging questions on the topic for the past several decades.

4.2 Vorticity Form of the Elsässer Variables

In this Section, we introduce evolution equations for the "vorticities" of the Elsässer variables. For want of better notation, define, in accord with (4.1), the curls of α and β by

$$X = \operatorname{curl}(\alpha) = \omega + j \qquad \text{and} \qquad Y = \operatorname{curl}(\beta) = \omega - j.$$

Then, as both α and β are divergence free, they can be recovered from X and Y by using the *Biot-Savart law*: assuming that the vector field f satisfies $\operatorname{div}(f) = 0$, we have

$$f_k \;=\; (-\Delta)^{-1} \sum_{j=1}^{d} \partial_j \big[\operatorname{curl}(f)\big]_{jk} \tag{4.2}$$

under very general far-field assumptions. For example, (4.2) holds for any $f \in \mathcal{S}'_h$. In the sequel, for the sake of simplicity, we will work with finite energy[3] solutions $\alpha, \beta \in L^\infty(L^2)$, so that there is no question as to whether (4.2) is valid or not. In addition, the good integrability properties of finite energy solutions will insure that the conditions of Theorem 3.11 apply, so the Elsässer system is equivalent to the original MHD equations. This means that we will use indiscriminately one or the other set of variables depending on what is most convenient.

Taking the curl of the Elsässer system in the previous chapter (3.2), we obtain two evolution equations for the quantities X and Y which read

$$\begin{cases} \partial_t X + (\beta \cdot \nabla) X = \mathcal{L}(\nabla\alpha, \nabla\beta) \\ \partial_t Y + (\alpha \cdot \nabla) Y = \mathcal{L}(\nabla\beta, \nabla\alpha). \end{cases} \tag{4.3}$$

In the above, $\mathcal{L}(\,.\,,\,.\,)$ is a matrix valued bilinear combination of the coefficients of the gradients $\nabla\alpha$ and $\nabla\beta$ which is analogous to the "stretching term" in the vorticity equation of 3D Euler equations. It is written

$$\big[\mathcal{L}(\nabla\alpha, \nabla\beta)\big]_{ij} \;=\; \partial_j\beta_k\,\partial_k\alpha_i \;-\; \partial_i\beta_k\,\partial_k\alpha_j\,,$$

where, as usual, there is an implicit sum on the repeated index $k = 1, ..., d$. In other words, the operator $\mathcal{L}$ is given by the (double of the) skew-symmetric part of a matrix product $\mathcal{L}(\nabla\alpha, \nabla\beta) = {}^t(\nabla\beta\,\nabla\alpha) - \nabla\beta\,\nabla\alpha$.

We conclude this Section by a few remarks concerning the case of plane fluids $d = 2$. The equations (4.3) are essentially transport equations, with an additional bilinear term $\mathcal{L}$ in the righthand side. It consequently is quite natural to ask if any of the usual methods for proving global well-posedness of the 2D Euler equations can be applied to plane MHD. It turns out that the presence of the righthand side is a source of considerable difficulty. For example, the methods of Yudovich (Chapter 8 in [81]) or Serfati [97] rely on L^p estimates for the scalar vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$. These are readily available for the 2D Euler system as ω is a solution of the pure transport equation

$$\partial_t\omega + u \cdot \nabla\omega = 0.$$

However, in the case of MHD[4], obtaining L^p estimates for the scalar quantities X and Y also requires bounding the bilinear term $\mathcal{L}(\nabla\alpha, \nabla\beta)$ in the space L^p, which can be done by writing

$$\big\|\mathcal{L}(\nabla\alpha, \nabla\beta)\big\|_{L^p} \lesssim \|\nabla\alpha\|_{L^p}\|\nabla\beta\|_{L^\infty}.$$

[3] A finite energy assumption is quite natural by virtue of the total energy balance equation (28) in the Introduction.

[4] This remark is true regardless of the dimension. But we will come back later to the special case $d = 2$ and point out some of its specific features.

The Biot-Savart law (4.2) shows that the derivatives $\nabla\alpha$ and $\nabla\beta$ are obtained from X and Y as the image of a Singular Integral Operator, and Calderón-Zygmund theory applies to give $\|\nabla\alpha\|_{L^p} \lesssim \|X\|_{L^p}$ provided that $1 < p < +\infty$ (see Theorem 1.23). However, this bound fails completely in the endpoint case $p = +\infty$, so at least one of the two factors in the inequality above remains troublesome.

This failure of simple L^p bounds indicates that we must turn to different function spaces to have any hope of closing the estimates: although subcritical Sobolev spaces $W^{m,p} \subsetneq W^{1,\infty}$ would provide *a priori* estimates on a finite time interval, we will rather work in Besov-Lipschitz spaces $\alpha, \beta \in B^s_{p,r} \subset W^{1,\infty}$. The reasons for this choice will appear clearly below. For now, we simply say that we will want to use the methods of Hmidi-Keraani [64] and Danchin-Fanelli [34] in order to find improved lower bounds for the lifespan of 2D solutions.

4.3 Existence of Solutions

In this Section, we show existence of solutions. For this, we implement a nowadays classical scheme. First of all, in Paragraph 4.3.1 we will show *a priori* estimates for smooth solutions in the relevant norms. From those estimates, we will also deduce a first lower bound (valid in any space dimension) on the lifespan of the solutions. After that, in Paragraph 4.3.2 we will give the explicit construction of smooth solutions to approximate problems, and show their convergence to a "true" solution of the original equations. The net result of these considerations is the following Theorem.

Theorem 4.1. *Let $s \in \mathbb{R}$ and $r \in [1, +\infty]$ such that $B^s_{\infty,r} \subset W^{1,\infty}$ and consider a set of divergence free initial data $(u_0, b_0) \in L^2 \cap B^s_{\infty,r}$. Then there exists a time $T > 0$ and weak solutions $(u, b) \in C^0([0, T[; B^s_{\infty,r} \cap L^2)$ of the ideal MHD system (3.1) associated to the initial (u_0, b_0), with the usual replacement of C^0 by C^0_w if $r = +\infty$. In addition, these solutions are associated to a (hydrodynamic) pressure force $\nabla\Pi = C^0([0, T[; B^s_{\infty,r} \cap L^2)$, with the same modification if $r = +\infty$.*

Remark 4.2. Originally, the methods we present in this chapter were developed in order to solve the following quasi-homogeneous MHD system

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + r\mathfrak{C}u + \nabla\pi = (b \cdot \nabla)b \\ \partial_t b + (u \cdot \nabla)b = (b \cdot \nabla)u \\ \partial_t r + u \cdot \nabla r = 0 \\ \mathrm{div}(u) = 0. \end{cases} \tag{4.4}$$

In (4.4) above, $r(t, x) \in \mathbb{R}$ represents a density perturbation function and $\mathfrak{C}$ is a constant $d \times d$ matrix. Equations (4.4) are a limit model for the 2D fast-rotating MHD system and will be derived as such in the next Chapter. It should be noted that all the results of this Chapter can in fact be applied to (4.4) with a few modifications. We refer to our treatment [29] of the subject.

Remark 4.3. The computations displayed below also lend themselves to a continuation criterion similar to those of the preceding Chapter: if

$$\int_0^T \Big\{ \|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} \Big\} \, \mathrm{d}t < +\infty,$$

then the solutions constructed in Theorem 4.1 can be extended beyond time T in the same space. However, the arguments are identical, and will be omitted here. It is notable that this continuation criterion also applies as such to the quasi-homogeneous system *without reference to the density perturbation r* (see [29]).

4.3.1 *A priori* Estimates

The main result of this part is stated in the next result, which contains basic *a priori* bounds for smooth solutions to system (3.1). For reference, we recall the ideal MHD system in Elsässer variables on which *a priori* estimates will be performed:

$$\begin{cases} \partial_t \alpha + (\beta \cdot \nabla)\alpha + \nabla \pi_1 = 0 \\ \partial_t \beta + (\alpha \cdot \nabla)\beta + \nabla \pi_2 = 0 \\ \mathrm{div}(\alpha) = \mathrm{div}(\beta) = 0. \end{cases} \tag{4.5}$$

Under good regularity and integrability assumptions, this system is equivalent to the vorticity formulation (4.3) and the Biot-Savart law (4.2).

Proposition 4.4. *Let (s,r) be such that $B^s_{\infty,r} \subset W^{1,\infty}$ or in other words, $s > 1$ or $s = r = 1$. Let (α, β) be regular solutions to the Elsässer system (4.5), related to regular initial data (α_0, β_0), with α_0 and β_0 being divergence free. Then, there exist a constant $C > 0$, which depends on the dimension d and (s,r), as well as a time $T^* > 0$, which depends on the above and $\big\|(\alpha_0, \beta_0)\big\|_{B^s_{\infty,r}}$, such that*

$$\big\|(\alpha(t), \beta(t))\big\|_{B^s_{\infty,r} \cap L^2} \leq C \big\|(\alpha_0, \beta_0)\big\|_{B^s_{\infty,r} \cap L^2}$$

for all $t \in [0, T^*]$. Moreover, we have the inequality

$$T^* \geq \frac{C}{\big\|(\alpha_0, \beta_0)\big\|_{B^s_{\infty,r} \cap L^2}}.$$

The rest of this paragraph is devoted to the proof of Proposition 4.4. Let us begin with a simple lemma, which allows to bound the bilinear expression $\mathcal{L}$ appearing in equations (4.3), and which we will treat as a forcing term.

Lemma 4.5. *Let (α, β) be a couple of functions in $B^s_{\infty,r}$, the vector fields α and β being divergence free. Then we have the following inequality:*

$$\big\|\mathcal{L}(\nabla\alpha, \nabla\beta)\big\|_{B^{s-1}_{\infty,r}} \lesssim \|\nabla \alpha\|_{L^\infty} \|\beta\|_{B^s_{\infty,r}} + \|\nabla \beta\|_{L^\infty} \|\alpha\|_{B^s_{\infty,r}}.$$

Proof of Lemma 4.5. This estimate immediately follows from the tame estimates if $s > 1$. The main difficulty is proving this last inequality for the endpoint case $s = r = 1$, namely for $\nabla\alpha$ and $\nabla\beta$ lying in $B^0_{\infty,1}$, which is not an algebra (see Section 1.5). To overcome the problem of working with a 0 regularity index, we use the fact that α and β are divergence free to rewrite things in the following way: by factoring the derivative ∂_k out of the product, we write $\mathcal{L}(\nabla\alpha, \nabla\beta)$ as the derivative of a $B^1_{\infty,1} \times B^0_{\infty,1}$ product:

$$\big[\mathcal{L}(\nabla\alpha, \nabla\beta)\big]_{ij} = \sum_{k=1}^d \Big(\partial_k\big(\alpha_i\, \partial_j \beta_k\big) - \partial_k\big(\alpha_j\, \partial_i \beta_k\big)\Big). \tag{4.6}$$

Now, making use of the Bony decomposition of a product, we get

$$\mathcal{L}(\nabla\alpha, \nabla\beta) = \mathcal{L}_{\mathcal{T}}(\nabla\alpha, \nabla\beta) + \mathcal{L}_{\mathcal{R}}(\nabla\alpha, \nabla\beta),$$

where we have defined

$$\big[\mathcal{L}_{\mathcal{T}}(\nabla\alpha, \nabla\beta)\big]_{ij} := \sum_{k=1}^d \Big(\mathcal{T}_{\partial_k \alpha_i}(\partial_j \beta_k) + \mathcal{T}_{\partial_j \beta_k}(\partial_k \alpha_i) - \mathcal{T}_{\partial_k \alpha_j}(\partial_i \beta_k) - \mathcal{T}_{\partial_i \beta_k}(\partial_k \alpha_j)\Big)$$

$$\big[\mathcal{L}_{\mathcal{R}}(\nabla\alpha, \nabla\beta)\big]_{ij} := \sum_{k=1}^d \Big(\mathcal{R}(\partial_k \alpha_i, \partial_j \beta_k) - \mathcal{R}(\partial_k \alpha_j, \partial_i \beta_k)\Big).$$

On the one hand, thanks to Proposition 1.20, we can easily estimate the paraproducts: with a little abuse of notation, we may write

$$\left\|\mathcal{T}_{\nabla\alpha}(\nabla\beta)\right\|_{B^0_{\infty,1}} + \left\|\mathcal{T}_{\nabla\beta}(\nabla\alpha)\right\|_{B^0_{\infty,1}} \lesssim \|\nabla\alpha\|_{L^\infty}\,\|\nabla\beta\|_{B^0_{\infty,1}} + \|\nabla\beta\|_{L^\infty}\,\|\nabla\alpha\|_{B^0_{\infty,1}}\,.$$

On the other hand, using equation (4.6), we can write the remainder terms in the following form:

$$\big[\mathcal{L}_{\mathcal{R}}(\nabla\alpha,\nabla\beta)\big]_{ij} = \sum_{k=1}^{d}\Big(\partial_k\mathcal{R}(\alpha_i,\partial_j\beta_k) - \partial_k\mathcal{R}(\alpha_j,\partial_i\beta_k)\Big)\,.$$

Now, each of the summands can be bounded thanks to Proposition 1.21. For instance, the first one is bounded by

$$\left\|\partial_k\mathcal{R}(\alpha_j,\partial_i\beta_k)\right\|_{B^0_{\infty,1}} \leq \left\|\mathcal{R}(\alpha_j,\partial_i\beta_k)\right\|_{B^1_{\infty,1}} \lesssim \|\nabla\alpha\|_{B^0_{\infty,\infty}}\|\beta\|_{B^1_{\infty,1}} \lesssim \|\nabla\alpha\|_{L^\infty}\|\beta\|_{B^1_{\infty,1}}\,,$$

where the last inequality is due to the embedding $L^\infty \hookrightarrow B^0_{\infty,\infty}$. The other summand can be dealt with in a symmetric way. Putting all this together, we finally get the sought bound for $\mathcal{L}(\nabla\alpha,\nabla\beta)$ in the space $B^0_{\infty,1}$. The lemma is thus proved. □

With the estimates of Lemma 4.5 at hand, we can turn to the proof of the Proposition.

Proof of Proposition 4.4. We start the proof by showing how the Biot-Savart law (4.2) allows us to find Besov estimates on the solution (α,β) based on the vorticities (X,Y). Assume the function $f \in B^s_{\infty,r}\cap L^2$ to be a divergence free vector field. From separating the low and high frequencies in the Biot-Savart law (4.2), we obtain

$$\|f\|_{B^s_{\infty,r}} \approx \sum_{j=1}^{d}\left\|\Delta_{-1}(-\Delta)^{-1}\sum_i\partial_i\big[\mathrm{curl}\,(f)\big]_{ij}\right\|_{L^\infty} + \left\|\mathbb{1}_{\{m\geq 0\}}\,2^{ms}\left\|\Delta_m(-\Delta)^{-1}\sum_i\partial_i\big[\mathrm{curl}\,(f)\big]_{ij}\right\|_{L^\infty}\right\|_{\ell^r(\nu\geq 0)}\,.$$

On the one hand, if $m\geq 0$, we know that $\Delta_m\big[\mathrm{curl}\,(f)\big]_{ij}$ is spectrally supported in an annulus, on which the symbol of the order -1 Fourier multiplier $(-\Delta)^{-1}\partial_i$ is smooth. Hence, by using the Bernstein inequalities of Lemma 1.7, we get

$$2^{sm}\left\|\Delta_m(-\Delta)^{-1}\sum_i\partial_i\big[\mathrm{curl}\,(f)\big]_{ij}\right\|_{L^\infty} \lesssim 2^{(s-1)m}\left\|\Delta_m\mathrm{curl}\,(f)\right\|_{L^\infty}\,.$$

On the other hand, using the fact that the symbol of $(-\Delta)^{-1}\nabla\mathrm{curl}$ is homogeneous of degree 0 and bounded on the unit sphere $|\xi|=1$, thus L^∞ in a neighborhood of the origin, Bernstein inequalities and Plancherel's theorem yield

$$\left\|\Delta_{-1}(-\Delta)^{-1}\sum_i\partial_i\big[\mathrm{curl}\,(f)\big]_{ij}\right\|_{L^\infty} \lesssim \left\|\Delta_{-1}(-\Delta)^{-1}\,\nabla\mathrm{curl}\,(f)\right\|_{L^2} \lesssim \|f\|_{L^2}\,.$$

Therefore, in the end, we deduce

$$\|f\|_{B^s_{\infty,r}} \lesssim \|f\|_{L^2} + \left\|\mathrm{curl}\,(f)\right\|_{B^{s-1}_{\infty,r}}\,. \tag{4.7}$$

Because the total energy of the solutions is conserved $\|(\alpha,\beta)\|_{L^2} = \|(\alpha_0,\beta_0)\|_{L^2}$, this bound tells us that, for bounding (α,β) in $B^s_{\infty,r}$, we can focus on estimates for the $B^{s-1}_{\infty,r}$ norms of X and Y, which solve system (4.3).

Since the quantities X and Y solve a system of transport equations (with $\mathcal{L}$ as a forcing term), we may apply the methods of the previous Chapter to find Besov estimates for the solutions. Let Δ_j be a dyadic block: applying it to system (4.3), we find

$$\begin{cases} \partial_t(\Delta_j X) + (\beta\cdot\nabla)\Delta_j X = \big[\beta\cdot\nabla, \Delta_j\big]X + \Delta_j\mathcal{L}(\nabla\alpha, \nabla\beta) \\ \partial_t(\Delta_j Y) + (\alpha\cdot\nabla)\Delta_j Y = \big[\alpha\cdot\nabla, \Delta_j\big]Y + \Delta_j\mathcal{L}(\nabla\beta, \nabla\alpha)\,. \end{cases}$$

Lemma 4.5 gives estimates in $B^{s-1}_{\infty,r}$ the terms on the far right:

$$\big\|\mathcal{L}(\nabla\alpha,\nabla\beta)\big\|_{B^{s-1}_{\infty,r}} \lesssim \|\alpha\|_{B^s_{\infty,r}}\|\beta\|_{B^s_{\infty,r}}\,.$$

On the other hand, for the commutator terms, we wish to use an estimate similar to that of Lemma 3.14. However, because we seek $B^s_{\infty,r}$ estimates on (α,β), the vorticities X and Y will only be bounded in $B^{s-1}_{\infty,r}$, which does not contain the space $W^{1,\infty}$ of Lipschitz functions if $s<2$, or $s=2$ and $r\neq 1$. In those cases, we must use a low regularity version of the Lemma.

Lemma 4.6. *Assume that $v\in B^s_{\infty,r}$ with $s>1$ or $s=r=1$, we have the inequality*

$$\forall\, f\in B^{s-1}_{\infty,r}\,,\qquad 2^{j(s-1)}\,\big\|\big[v\cdot\nabla,\Delta_j\big]f\big\|_{L^\infty} \lesssim c_j\Big(\|\nabla v\|_{L^\infty}\|f\|_{B^{s-1}_{\infty,r}} + \|\nabla v\|_{B^{s-1}_{\infty,r}}\|f\|_{L^\infty}\Big)\,,$$

where $\big(c_j\big)_{j\geq -1}$ is a sequence in the unit ball of ℓ^r.

In those cases, we must use the inequality of Lemma 4.6 instead of Lemma 3.14. In both situations, however, we get the bound

$$2^{j(s-1)}\Big(\,\big\|\big[\beta\cdot\nabla,\Delta_j\big]X\big\|_{L^\infty} + \big\|\big[\alpha\cdot\nabla,\Delta_j\big]Y\big\|_{L^\infty}\Big) \lesssim c_j(t)\|\alpha\|_{B^s_{\infty,r}}\|\beta\|_{B^s_{\infty,r}}\,, \tag{4.8}$$

for a suitable sequence $\big(c_j(t)\big)_{j\geq -1}$ belonging to the unit sphere of ℓ^r. Using all this to write an L^∞ estimate for $\Delta_j(X,Y)$, we get

$$2^{j(s-1)}\big\|\Delta_j\big(X(t),Y(t)\big)\big\|_{L^\infty} \lesssim 2^{j(s-1)}\big\|\Delta_j(X_0,Y_0)\big\|_{L^\infty} + \int_0^t c_j(\tau)\|\alpha\|_{B^s_{\infty,r}}\|\beta\|_{B^s_{\infty,r}}\mathrm{d}\tau\,, \tag{4.9}$$

At this point, for all $t\geq 0$, we note $E(t)$ the quantity on which estimates are sought:

$$E(t) := \big\|\big(\alpha(t),\beta(t)\big)\big\|_{L^2} + \big\|\big(X(t),Y(t)\big)\big\|_{B^{s-1}_{\infty,r}}\,.$$

Using the conservation of energy and the previous inequality (4.9), we get, thanks to (4.7) and the Minkowski inequality (see Proposition 1.3 in [7]), the bound

$$E(t) \lesssim E(0) + \int_0^t E(\tau)^2\,\mathrm{d}\tau\,.$$

To end the proof, we define the time $T^*>0$ by

$$T^* = \sup\left\{T>0\;\middle|\;\int_0^t E(\tau)^2\,\mathrm{d}\tau \leq E(0)\right\}.$$

Then we deduce $E(t)\leq CE(0)$ for all times $t\in[0,T^*]$ and for some positive constant $C=C(d,s,r)$. Therefore, for such times, the following inequality holds:

$$\int_0^t E(\tau)^2\,\mathrm{d}\tau \leq tE(0)^2.$$

By using the definition of T^*, we see that

$$T^* \geq \frac{C}{E(0)}\,,$$

for a suitable positive constants C. This ends the proof of the proposition. □

4.3.2 Proof of Existence

In the previous paragraph, we have shown *a priori* bounds, in the relevant norms, for smooth solutions to system (4.5). Here, we present the proof of the existence of solutions at the claimed level of regularity.

For this, we follow a standard procedure: first of all, we construct a sequence of smooth solutions to approximate problems. Next, from the estimates of Paragraph 4.3.1 we deduce uniform bounds for that sequence of approximate solutions. Finally, by use of those uniform bounds and an energy argument, we are able to show strong convergence properties for suitable quantities, which in turn allow us to take the limit in the approximation parameter and gather the existence of a solution to the original problem. For the sake of simplicity, we are going to assume $r < +\infty$: the case $r = +\infty$ can be handled with minor modifications.

Construction of smooth approximate solutions. For any $n \in \mathbb{N}$, let us define

$$\big(\alpha_0^n\,,\,\beta_0^n\big)\;:=\;\big(S_n\alpha_0\,,\,S_n\beta_0\big)\,,$$

where $S_n = \sum_{m\leq n-1}\Delta_m$ is the low frequency cut-off operator introduced in Remark 1.9. By the finite energy assumption $\alpha_0, \beta_0 \in L^2$, one has, for any $n \in \mathbb{N}$, $\alpha_0^n, \beta_0^n \in H^\infty := \bigcap_{\sigma\in\mathbb{R}} H^\sigma$, which is obviously embedded (for a suitable topology on H^∞) in the space C_b^∞ of C^∞ functions which are globally bounded together with all their derivatives. In addition, we have

$$\big(\alpha_0^n\,,\,\beta_0^n\big)\;\underset{n\to+\infty}{\longrightarrow}\;\big(\alpha_0\,,\,\beta_0\big)\quad\text{ in }\;L^2\cap B^s_{\infty,r}\,. \tag{4.10}$$

This having been done, we are going to define a sequence of approximate solutions to system (3.1) by induction. First of all, we set $\big(\alpha^0\,,\,\beta^0\big) := \big(\alpha_0^0\,,\,\beta_0^0\big)$. Obviously, for all $\sigma \in \mathbb{R}$, we have that $\alpha^0\,,\,\beta^0\;\in C^0\big(\mathbb{R}_+;H^\sigma\big)$, with $\mathrm{div}\,(\alpha^0) = \mathrm{div}\,(\beta^0) = 0$. Next, assume that the triplet $\big(R^n, \alpha^n, \beta^n\big)$ is given, with, for all $\sigma \in \mathbb{R}$, the properties

$$\alpha^n\,,\,\beta^n\;\in\;C^0\big(\mathbb{R}_+;H^\sigma\big)\qquad\text{and}\qquad\mathrm{div}\,(\alpha^n)\;=\;\mathrm{div}\,(\beta^n)\;=\;0\,.$$

First of all, we solve the two (linear) transport equations with divergence free constraints

$$\begin{cases}\partial_t\,\alpha^{n+1}\,+\,(\beta^n\cdot\nabla)\alpha^{n+1}\,+\,\nabla\pi_1^{n+1}\,=0\\ \partial_t\,\beta^{n+1}\,+\,(\alpha^n\cdot\nabla)\beta^{n+1}\,+\,\nabla\pi_2^{n+1}\,=0\\ \mathrm{div}\,\big(\alpha^{n+1}\big)\;=\;\mathrm{div}\,\big(\beta^{n+1}\big)\;=\;0\,,\end{cases} \tag{4.11}$$

with initial data $\alpha^{n+1}_{|t=0} = \alpha_0^{n+1}$ and $\beta^{n+1}_{|t=0} = \beta_0^{n+1}$, to define the vector fields α^{n+1} and β^{n+1}. It is not hard to solve the previous linear problem by energy methods; see also Propositions 3.2 and 3.4 of [33] in this respect. We thus find unique solutions α^{n+1} and β^{n+1}, belonging to the space $C^0\big(\mathbb{R}_+;H^\sigma\big)$ for all $\sigma\in\mathbb{R}$.

We omit here the analysis of the pressure gradients $\nabla\pi_1^{n+1}$ and $\nabla\pi_2^{n+1}$ (which are present to restore the divergence free conditions on α^{n+1} and β^{n+1}), since they are not needed in the rest of the present proof. However, this analysis can be performed following the argument we will use in the last paragraph of this section, in order to establish the regularity of the (limit) pressure functions $\nabla\pi_1$ and $\nabla\pi_2$.

Uniform bounds for the approximate solutions. We now have to show uniform bounds for the sequence $\big(\alpha^n, \beta^n\big)_{n\in\mathbb{N}}$ we have constructed above. We argue by induction, and prove that there exists a time $T > 0$ such that the following property holds true: for all $t \in [0, T]$ and all $n \in \mathbb{N}$, one has

$$\big\|\big(\alpha^n(t)\,,\,\beta^n(t)\big)\big\|_{L^2}\;\leq\;C\;\|(\alpha_0\,,\,\beta_0)\|_{L^2} \tag{4.12}$$

$$\left\|\left(\alpha^n(t),\beta^n(t)\right)\right\|_{L^2\cap B^s_{\infty,r}} \leq C\,K_0, \tag{4.13}$$

where the constant $C>0$ does not depend on the data or the solutions, and therefore not on n, and where we have set $K_0 := \|(\alpha_0,\beta_0)\|_{L^2\cap B^s_{\infty,r}}$. It is obvious that the initial couple $\left(\alpha^0,\beta^0\right)$ satisfies the previous requirements. Assume now that, for some $n\in\mathbb{N}$, the couple $\left(\alpha^n,\beta^n\right)$ verifies the same conditions on some time interval $[0,T]$. We want to prove that, in the same time interval, $\left(\alpha^{n+1},\beta^{n+1}\right)$ also enjoys the same properties. First of all, a straightforward computation shows that the conservation of energy also holds for the approximate solutions, so that $\|(\alpha^{n+1},\beta^{n+1})\|_{L^2} = \|(\alpha_0^{n+1},\beta_0^{n+1})\|_{L^2}$. In order to get bounds for the Besov norms, we resort to the vorticity formulation of (4.11): applying the curl operator to that system leads us to

$$\begin{cases} \partial_t X^{n+1} + (\beta^n\cdot\nabla)X^{n+1} = \mathcal{L}\left(\nabla\alpha^{n+1},\nabla\beta^n\right) \\ \partial_t Y^{n+1} + (\alpha^n\cdot\nabla)Y^{n+1} = \mathcal{L}\left(\nabla\beta^{n+1},\nabla\alpha^n\right). \end{cases} \tag{4.14}$$

Proceeding exactly as in the proof of Proposition 4.4, but for equation (4.14), we find an estimate analogous to (4.9): for any $t\geq 0$ one has

$$\begin{aligned} 2^{j(s-1)}\left\|\Delta_j\left(X^{n+1}(t),Y^{n+1}(t)\right)\right\|_{L^\infty} &\lesssim 2^{j(s-1)}\left\|\Delta_j\left(X_0^{n+1},Y_0^{n+1}\right)\right\|_{L^\infty} \\ &\quad + \int_0^t c_j(\tau)\left\|\left(\alpha^{n+1},\beta^{n+1}\right)\right\|_{B^s_{\infty,r}}\left\|\left(\alpha^n,\beta^n\right)\right\|_{B^s_{\infty,r}}\,\mathrm{d}\tau, \end{aligned} \tag{4.15}$$

where, as usual, the sequence $\left(c_j(t)\right)_j$ belongs to the unit sphere of ℓ^r. At this point, as in the proof of Proposition 4.4, we define $E(t)$ to be the $B^s_{\infty,r}\cap L^2$ norm of $(\alpha(t),\beta(t))$ for all $t\geq 0$. By (4.7), we have

$$\begin{aligned} E^{n+1}(t) &:= \left\|\left(\alpha^{n+1}(t),\beta^{n+1}(t)\right)\right\|_{L^2\cap B^s_{\infty,r}} \\ &\approx \left\|\left(\alpha^{n+1}(t),\beta^{n+1}(t)\right)\right\|_{L^2} + \left\|\mathrm{curl}\left(\alpha^{n+1}(t),\beta^{n+1}(t)\right)\right\|_{B^{s-1}_{\infty,r}}. \end{aligned} \tag{4.16}$$

Thus, taking the ℓ^r norm in (4.15) and using the conservation of energy (4.12) at level $n+1$, we obtain

$$E^{n+1}(t) \leq C\left(E^{n+1}(0) + \int_0^t E^{n+1}(\tau)\left\|\left(\alpha^n(\tau),\beta^n(\tau)\right)\right\|_{B^s_{\infty,r}}\,\mathrm{d}\tau\right).$$

An application of Grönwall's lemma and the fact that $E^{n+1}(0)\lesssim K_0$ gives

$$E^{n+1}(t) \leq C\,K_0\,\exp\left(C\int_0^t \left\|\left(\alpha^n(\tau),\beta^n(\tau)\right)\right\|_{B^s_{\infty,r}}\,\mathrm{d}\tau\right). \tag{4.17}$$

Our inductive assumption (4.13) provides, after integration of the previous inequality,

$$\int_0^t \left\|\left(\alpha^n(\tau),\beta^n(\tau)\right)\right\|_{B^s_{\infty,r}}\,\mathrm{d}\tau \leq \frac{C}{c}\left(e^{c\,K_0\,t}-1\right).$$

Observe that, for $0\leq x\leq 1$, one has $e^x-1\leq x+x^2\leq 2x$. So, if $T>0$ is chosen so small that $cK_0T\leq 1$, we finally deduce from the previous bound and (4.17) that

$$E^{n+1}(t) \leq C\,K_0\,\exp\left(\frac{C}{c}\left(e^{c\,K_0\,t}-1\right)\right) \leq C\,K_0\,e^{2CK_0\,t},$$

completing in this way the proof of (4.13) at the level $n+1$.

Convergence. It remains us to show convergence of the sequence $\big(\alpha^n,\beta^n\big)_n$ towards a solution (α,β) of the original problem (3.1): this is our next goal. By an energy method similar to the one we will use later for proving uniqueness, we are going to show that $\big(\alpha^n\big)_n$ and $\big(\beta^n\big)_n$ are Cauchy sequences in the space $C^0\big([0,T];L^2\big)$. For this, we introduce the following notation: for any couple $(n,p)\in\mathbb{N}^2$, we define the quantities

$$\delta\alpha^{n,p}\,:=\,\alpha^{n+p}-\alpha^n\qquad\text{and}\qquad\delta\beta^{n,p}\,:=\,\beta^{n+p}-\beta^n\,.$$

Of course, $\mathrm{div}\,(\delta\alpha^{n,p})\,=\,\mathrm{div}\,(\delta\beta^{n,p})\,=\,0$ for any $(n,p)\in\mathbb{N}^2$. In addition, after setting $\delta\pi_j^{n,p}:=\pi_j^{n+p}-\pi_j^n$ for $j=1,2$, simple computations yield the system of equations

$$\begin{cases}\partial_t\delta\alpha^{n,p}+(\beta^{n+p-1}\cdot\nabla)\delta\alpha^{n,p}+\nabla\delta\pi_1^{n,p}=-(\delta\beta^{n-1,p}\cdot\nabla)\alpha^n\\ \partial_t\delta\beta^{n,p}+(\alpha^{n+p-1}\cdot\nabla)\delta\beta^{n,p}+\nabla\delta\pi_2^{n,p}=-(\delta\alpha^{n-1,p}\cdot\nabla)\beta^n\,,\end{cases}\tag{4.18}$$

supplemented with initial data $\big(\delta\alpha^{n,p},\delta\beta^{n,p}\big)_{|t=0}\;=\;\big(\delta\alpha^{n,p}(0),\delta\beta^{n,p}(0)\big)$. Then we perform an energy estimate on this system: by testing the first equation in (4.18) by $\delta\alpha^{n,p}$ and the second one by $\delta\beta^{n,p}$, we get

$$\left\|\big(\delta\alpha,\delta\beta\big)^{n,p}(t)\right\|_{L^2}\,\leq\,\left\|\big(\delta\alpha,\delta\beta\big)^{n,p}(0)\right\|_{L^2}+C\int_0^t\left\|\big(\delta\alpha,\delta\beta\big)^{n-1,p}\right\|_{L^2}\left\|\big(\nabla\alpha^n,\nabla\beta^n\big)\right\|_{L^\infty}\,\mathrm{d}t.$$

By using the uniform bounds (4.12) and (4.13) on the approximate solutions, as well as Besov-Lipschitz embeddings $B^s_{\infty,r}\subset W^{1,\infty}$, we know that

$$\sup_{t\in[0,T]}\left\|\nabla\big(\alpha^n,\beta^n\big)(t)\right\|_{L^\infty}+\int_0^T\Big(\left\|\alpha^{n+p-1},\beta^{n+p-1}\right\|_{L^2}+\left\|\big(\alpha^{n-1},\beta^{n-1}\big)\right\|_{L^2\cap L^\infty}\Big)\,\mathrm{d}t\,\leq\,C_T\,,$$

for a constant C_T depending on T, but uniform with respect to n and p. Therefore, from the previous inequalities and Grönwall's lemma, we deduce

$$\sup_{[0,t]}\left\|\big(\delta\alpha,\delta\beta\big)^{n,p}\right\|_{L^2}\,\leq\,C_T\left(\left\|\big(\delta\alpha,\delta\beta\big)^{n,p}(0)\right\|_{L^2}+\int_0^t\sup_{[0,\tau]}\left\|\big(\delta\alpha,\delta\beta\big)^{n-1,p}\right\|_{L^2}\mathrm{d}\tau\right).$$

After setting

$$F^n(t)\,:=\,\sup_{p\geq0}\sup_{[0,t]}\left\|\big(\delta\alpha,\delta\beta\big)^{n,p}\right\|_{L^2}\qquad\text{and}\qquad D_0^n\,:=\,\sup_{p\geq0}\left\|\big(\delta\alpha,\delta\beta\big)^{n,p}(0)\right\|_{L^2}\,,$$

the previous estimate implies that, for all $t\in[0,T]$, one has

$$F^n(t)\,\leq\,C_T\,D_0^n\,+\,C_T\int_0^tF^{n-1}(\tau)\,\mathrm{d}\tau\,.\tag{4.19}$$

A simple induction argument yields, for all $t\in[0,T]$, the bound

$$F^n(t)\,\leq\,C_T\sum_{k=0}^{n-1}\left(\frac{(C_T\,T)^k}{k!}\,D_0^{n-k}\right)\,+\,\frac{(C_T\,T)^n}{n!}\,F^0(t)\,.$$

This having been established, we notice that, owing to (4.10), we have that

$$\lim_{n\to+\infty}\sup_{p\geq0}\left\|\big(\delta\alpha,\delta\beta\big)^{n,p}(0)\right\|_{L^2}\,=\,0\,.$$

Hence, using dominated convergence, we can take the limit for $n\to+\infty$ in (4.19) and conclude, thanks to Grönwall's lemma, that

$$\lim_{n\to+\infty}\sup_{p\geq0}\sup_{t\in[0,T]}\left\|\big(\delta\alpha,\delta\beta\big)^{n,p}(t)\right\|_{L^2}\,=\,0\,.$$

This property implies that $\left(\alpha^n\right)_n$ and $\left(\beta^n\right)_n$ are Cauchy sequences in $C^0\big([0,T];L^2\big)$, thus they converge respectively to some α and β in that space.

Observe that, owing to the embedding $L^2 \hookrightarrow B^{-d/2}_{\infty,2}$, to uniform bounds and to interpolation, the sequences $\left(\alpha^n\right)_n$ and $\left(\beta^n\right)_n$ also strongly converge in any intermediate space $L^\infty_T(B^\sigma_{\infty,r})$, with $\sigma < s$, and in particular in $L^\infty([0,T]\times\mathbb{R}^d)$. Thus, it is easy to pass to the limit in the weak formulation of equation (4.11), finding that the couple (α,β) is a weak solution to the original problem (4.5), for suitable pressure gradients $\nabla\pi_1$ and $\nabla\pi_2$. Space regularity for $\big(R,\alpha,\beta\big)$ in $B^s_{\infty,r}$ follows by uniform bounds and Fatou's property in Besov spaces. By the analysis preformed in the proof of Theorem 3.11, we also know that $\nabla\pi_1 = \nabla\pi_2$.

Regularity of the pressure terms, and final checks. Let us now devote some attention to the study of the regularity of $\nabla\pi_1$. First of all, similar computations as the ones leading to (4.7) give the bound

$$\left\|\nabla\pi_1\right\|_{L^2\cap B^s_{\infty,r}} \lesssim \left\|\nabla\pi_1\right\|_{L^2} + \left\|\Delta\pi_1\right\|_{B^{s-1}_{\infty,r}} . \tag{4.20}$$

Now, applying the div operator to the first equation in (3.1), we deduce that π_1 satisfies the elliptic equation

$$-\Delta\pi_1 = \operatorname{div} F\,, \qquad \text{where} \qquad F := (\beta\cdot\nabla)\alpha\,. \tag{4.21}$$

On the one hand, an application of the Lax-Milgram theorem implies that

$$\left\|\nabla\pi_1\right\|_{L^2} \lesssim \left\|F\right\|_{L^2} \lesssim \left\|\beta\right\|_{L^2}\left\|\nabla\alpha\right\|_{L^\infty}\,,$$

so that $\nabla\pi_1 \in L^\infty_T(L^2)$. On the other hand, we observe that, owing to the divergence free condition on α and β, one has $\operatorname{div}\big((\beta\cdot\nabla)\alpha\big) = \nabla\beta : \nabla\alpha = \sum_{j,k}\partial_j\beta_k\,\partial_k\alpha_j$. Therefore,

$$\left\|\Delta\pi_1\right\|_{B^{s-1}_{\infty,r}} \lesssim \left\|\nabla\beta:\nabla\alpha\right\|_{B^{s-1}_{\infty,r}} \lesssim \left\|\beta\right\|_{B^s_{\infty,r}}\left\|\alpha\right\|_{B^s_{\infty,r}}\,.$$

Notice that, when $s=1$, the estimate $\left\|\nabla\beta:\nabla\alpha\right\|_{B^{s-1}_{\infty,r}} \lesssim \left\|\beta\right\|_{B^s_{\infty,r}}\left\|\alpha\right\|_{B^s_{\infty,r}}$ still holds. For proving this, one has to argue as in the proof of Corollary 3.19, and use the divergence free condition on α (or β) in order to bound the remainders appearing in the Bony decomposition of the previous product $\nabla\beta:\nabla\alpha$. In the end, we deduce that $\Delta\pi_1$ belongs to $L^\infty_T(B^{s-1}_{\infty,r})$. Thus, from (4.20) and those two pieces of information, we conclude that $\nabla\pi_1 \in L^\infty_T(L^2\cap B^s_{\infty,1})$.

This having been established, we can use classical results on solutions to transport equations in Besov spaces (see Theorem 3.19 in [7]) to infer the claimed time continuity of α and β with values in $B^s_{\infty,r}\cap L^2$, and that the pressure $\nabla\pi_1$ belongs to $C^0_T(L^2\cap\dot B^s_{p,r})$. Finally, the claimed regularity properties for the time derivatives $\partial_t u$ and $\partial_t b$ follow from an inspection of the equations in (4.5). The proof of the existence is now completed.

4.4 Uniqueness by an Energy Method

In this section, we focus on the uniqueness of solutions. As we have explained in Chapter 3, a necessary condition for the ideal MHD system (4.5) to be well-posed consists in requiring the solutions to have some integrability property at infinity, whereas there is no hope of uniqueness for solutions that are solely bounded. In our framework, this is guaranteed by the finite-energy condition on the initial data.

Uniqueness in our functional framework is a straightforward consequence of the following stability result, whose proof is based on an energy method.

Theorem 4.7. *Let (u_1,b_1) and (u_2,b_2) be two solutions*[5] *to the ideal MHD system (4.5). Assume that, for some $T>0$, one has the following properties:*

[5] To fix ideas, say weak solutions in the sense of Definition 3.8 in the previous chapter.

(i) the two quantities $\delta u := u_1 - u_2$ *and* $\delta b := b_1 - b_2$ *belong to the space* $C^1\big([0,T];L^2(\mathbb{R}^d)\big)$*;*

(ii) $\nabla u_1, \nabla b_1 \in L^1\big([0,T];L^\infty(\mathbb{R}^d)\big)$*;*

Then, for all $t \in [0,T]$*, we have the stability inequality*

$$\big\|(\delta u,\delta b)(t)\big\|_{L^2} \,\leq\, C\,\big\|(\delta u,\delta b)(0)\big\|_{L^2}\,e^{CA(t)}\,,$$

for a universal constant $C>0$*, where we have defined*

$$A(t) \,:=\, \int_0^t \Big\{\|\nabla u_1(\tau)\|_{L^\infty} + \|\nabla b_1(\tau)\|_{L^\infty}\Big\}\mathrm{d}\tau\,.$$

Remark 4.8. Employing similar arguments as the ones used in [28] (see the proof to Theorem 4.3 therein, or equally the proof of the relative entropy of Subsection 5.3.2), it would be enough to assume $C^0_T(L^2)$ regularity for $\delta\alpha$ and $\delta\beta$. In that case, the previous theorem would become a full-fledged weak-strong uniqueness result. For the sake of simplicity, we do not pursue that issue here, and we assume that $\delta\alpha$ and $\delta\beta$ belong to $C^1_T(L^2)$.

Proof. The claimed bound is simply based on energy estimates for the difference of the two solutions. The original MHD system (3.1) is symmetric, so one could implement that strategy directly on the (R,u,b)-formulation of the equations. However, in order to avoid unpleasant derivatives on those differences (which we do not know to be smooth enough), it is better to work in Elsässer variables.

Therefore, with obvious notations, let us introduce the Elsässer variables (α_1,β_1) and (α_2,β_2), which solve system (4.5). This is possible thanks to item *1* in Theorem 3.11; notice that this step is based only on algebraic manipulations of the equations, and requires no special integrability condition on the magnetic field. Set

$$\delta\alpha \,:=\, \alpha_1 \,-\, \alpha_2 \qquad\qquad \text{and} \qquad\qquad \delta\beta \,:=\, \beta_1 \,-\, \beta_2\,.$$

We take the difference of the two systems solved by (α_1,β_1) and (α_2,β_2) to obtain

$$\begin{cases} \partial_t(\delta\alpha) + (\beta_2\cdot\nabla)\delta\alpha + (\delta\beta\cdot\nabla)\alpha_1 + \nabla\delta\pi_1 = 0 \\ \partial_t(\delta\beta) + (\alpha_2\cdot\nabla)\delta\beta + (\delta\alpha\cdot\nabla)\beta_1 + \nabla\delta\pi_2 = 0 \\ \mathrm{div}(\delta\alpha) \,=\, \mathrm{div}(\delta\beta) \,=\, 0\,, \end{cases} \tag{4.22}$$

where we have denoted $\delta\pi_1$ and $\delta\pi_2$ the difference of the two pressure terms appearing in system (4.5) and related to the solution couples (α_1,β_1) and (α_2,β_2). We start by testing the first equation against $\delta\alpha$. Owing to the divergence free conditions on $\delta\alpha$ and $\delta\beta$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\delta\alpha\|^2_{L^2} = -\int(\delta\beta\cdot\nabla)\alpha_1\cdot\delta\alpha\,\mathrm{d}x\,.$$

Bounding the integral on the right-hand side of the previous equality is fairly easy: after using the Cauchy-Schwarz and Young inequalities, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\delta\alpha\|^2_{L^2} \,\leq\, \|\nabla\alpha_1\|_{L^\infty}\,\big\|(\delta\alpha,\delta\beta)\big\|^2_{L^2}\,,$$

Performing, *mutatis mutandi*, the same computations with the second equation, we find an analogous inequality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\delta\beta\|^2_{L^2} \,\leq\, \|\nabla\beta_1\|_{L^\infty}\,\big\|(\delta\alpha,\delta\beta)\big\|^2_{L^2}\,.$$

Putting both those inequalities together leads us to a final inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|(\delta\alpha,\delta\beta)\big\|^2_{L^2} \leq \Big\{\|\nabla u_1\|_{L^\infty} + \|\nabla b_1\|_{L^\infty}\Big\}\,\big\|(\delta R,\delta\alpha,\delta\beta)\big\|^2_{L^2}\,.$$

on which an application of Grönwall's lemma ends the proof. □

From the previous result, it is possible to deduce uniqueness of solutions in the considered functional framework. We summarize this in the following Proposition.

Proposition 4.9. *Let $s \in \mathbb{R}$ such that $s > 1$ or $s = r = 1$ and consider an initial data couple $(u_0, b_0) \in B^s_{\infty,r} \cap L^2$. For any time $T > 0$, there exists at most one solution $(u, b) \in C^0([0, T[; B^s_{\infty,r} \cap L^2)$ related to those initial data.*

Proof. Let us consider an initial datum (u_0, b_0) satisfying the assumptions stated above. Let (α_1, β_1) and (α_2, β_2) be two associated solutions to the Elsässer system (4.5) related to that initial datum.

It is not hard to see that all the assumptions made in Theorem 4.7 are matched by those solutions. The only point which deserves some explanation is Condition (i): let us give some details. We focus on the regularity of the quantity $\delta\alpha$, the proof being identical for $\delta\beta$.

Let us focus on the first equation of (4.22): the function $\delta\alpha$ takes the value $\delta\alpha_{|t=0} = 0$ at initial time and is transported by a divergence free vector field, under the action of the "external force"

$$f = -\big[\beta_2 \cdot \nabla,\, \mathbb{P}\big]\delta\alpha - \mathbb{P}(\delta\beta \cdot \nabla)\alpha_1.$$

By assumption, and since the Leray projector is bounded $\mathbb{P} : L^2 \longrightarrow L^2$, we infer that our forcing term lies in the space $f \in C^0\big([0, T]; L^2\big)$. Therefore, by standard theory of transport equations (or Theorem 3.19 in [7] applied to $L^2 = B^0_{2,2}$) we get $\delta\alpha \in C^0\big([0, T]; L^2\big)$. By the same token, we also see that $\partial_t\delta\alpha$ belongs to the same space, so finally $\delta\alpha \in C^1\big([0, T]; L^2\big)$, as claimed.

In the end, we can apply Theorem 4.7 to deduce that $\big\|(\delta u, \delta b)\big\|_{L^\infty_T(L^2)} = 0$. By both time and space continuity, we infer that $\big(u_1, b_1\big)(t, x) = \big(u_2, b_2\big)(t, x)$ for *all* $(t, x) \in [0, T] \times \mathbb{R}^d$. This means exactly the sought uniqueness. □

4.5 Improved Lifespan in the Two-Dimensional Case

In this Section, we focus on the case of plane MHD and attempt to use the vorticity form of the equations to find an improved lifespan for solutions as in Theorem 3.29. Contrary to the method we developed in the previous chapter, we will make use of some of the specific features of the equations in dimension $d = 2$.

In all that follows, we consider the case of space dimension $d = 2$. Thus, the curl of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ can be identified with the scalar function

$$\mathrm{curl}\,(f) = \partial_1 f_2 - \partial_2 f_1\,.$$

4.5.1 The structure of the bilinear term

In this paragraph, we focus on the structure and properties of the bilinear term $\mathcal{L}(\nabla\alpha, \nabla\beta)$. Just as the curl matrix, $\mathcal{L}$ lies in vector line of skew-symmetric 2×2 matrices, and can therefore be represented by a scalar function:

$$\mathcal{L}(\nabla\alpha, \nabla\beta) = \partial_1\alpha_1\ (\partial_1\beta_2 + \partial_2\beta_1)\ +\ \partial_2\beta_2\ (\partial_1\alpha_2 + \partial_2\alpha_1)\ . \tag{4.23}$$

By noting $\mathfrak{T}_2$ the linear space of traceless 2×2 matrices, we may see $\mathcal{L}$ as a bilinear operator $\mathcal{L} : \mathfrak{T}_2 \times \mathfrak{T}_2 \longrightarrow \mathbb{R}$ which is, by virtue of (4.23), and in the special case $d = 2$, skew-symmetric. This is expected, as the curl of a transport-type term $f_k\partial_k g$ is, for $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$\mathrm{curl}\,\big((f \cdot \nabla)g\big) = f \cdot \nabla\mathrm{curl}\,(g) - \mathcal{L}(\nabla f, \nabla g)$$

and must reduce to $f \cdot \nabla\mathrm{curl}\,(g)$ when $f = g$. This is the reason why the vorticity solves a pure transport equation in the case of 2D Euler equations, and is at the heart of all global well-posedness results for this last system.

Because we expect $\mathcal{L}$ to take a simple form when there is no magnetic field $b \equiv 0$, in which case the equations are reduced to the Euler system, we attempt to write $\mathcal{L}$ as a function of the physical variables u and b. By using the skew-symmetry, which implies that $\mathcal{L}(\nabla f, \nabla f) = 0$ for any divergence free vector field f, we get

$$\mathcal{L}(\nabla\alpha, \nabla\beta) = \mathcal{L}\big(\nabla(u+b), \nabla(u-b)\big) = -2\,\mathcal{L}(\nabla u, \nabla b)\,. \tag{4.24}$$

More importantly for what follows, the bilinear term $\mathcal{L}(\nabla\alpha, \nabla\beta)$ will be bounded linearly with respect to the magnetic field, so that the vorticity equations (4.3) will nearly be pure transport as the magnetic field is small.

4.5.2 Improved Lifespan for the Solutions

In this paragraph, we present our original proof of Theorem 3.29 which relies on the vorticity form of the Elsässer system. The argument is inspired by the global *a priori* estimates of Hmidi and Keraani [64] for the 2D Euler equations (which can also be found in paragraph 7.2.2 of [7]). In order to attain a $B^1_{\infty,1}$ regularity of the solution, which is enough te provide uniqueness of solutions, the authors of [64] seek instead $B^0_{\infty,1}$ *a priori* bounds on the vorticity ω. But since ω is a solution of the pure transport equation

$$\partial_t\omega + u\cdot\nabla\omega = 0,$$

it is possible to find bounds for $\|\omega\|_{B^0_{\infty,1}}$ that are linear with respect to the solution, and hence provide global *a priori* estimates (see Subsection 3.5.1 for details of the argument). This linear estimate for solutions of pure transport equations is a sort of logarithmic interpolation between the conservation of $\|\omega\|_{L^\infty}$ and usual exponential bounds one obtains in higher order spaces.

In the context of ideal MHD, if we want to find $B^1_{\infty,1}$ solutions, we have at our disposal order zero quantities (X, Y), which solve a system of transport equations,[6] but with extra righthand side $\mathcal{L}$ terms. However, in a low magnetic field regime, the computations (4.24) above show that the $\mathcal{L}$ terms should be of small size, so we may hope for an improved well-posedness theory. These methods have been used in [34] and [43], and, more recently, in [94]. The sum of these considerations is the following Theorem.

Theorem 4.10. *Assume that $d = 2$. Consider a set of initial data $(u_0, b_0) \in L^2 \cap B^2_{\infty,1}$ such that $\mathrm{div}(u_0) = \mathrm{div}(b_0) = 0$ and let $(u, b) \in C^0([0, T[; L^2B^2_{\infty,1})$ be the unique solution of the ideal MHD system (4.5) provided by Theorems 4.1 and 4.7. Then the lifespan T of this solution can be taken such that*

$$T \geq \frac{C}{\|(u_0, b_0)\|_{B^2_{\infty,1}\cap L^2}} \log\left\{1 + \log\left[1 + \log\left(1 + C\frac{\|(u_0, b_0)\|_{B^1_{\infty,1}\cap L^2}}{\|b_0\|_{B^1_{\infty,1}}}\right)\right]\right\}.$$

Remark 4.11. Owing to the bilinearity of $\mathcal{L}$, a result similar to Theorem 4.10 holds true for solutions to the Elsässer system (4.5) with respect to small values of β, *regardless the space dimension.* Moreover, as we have a better equation for β in (3.1) then we have for b, we have no need of initial data in the higher regularity space $B^2_{\infty,1}$ as in Theorems 4.10 and 3.29. Specifically, we have in all dimensions:

$$T \geq \frac{C}{\big\|(\alpha_0, \beta_0)\big\|_{B^1_{\infty,1}\cap L^2}} \log\left\{1 + \log\left(1 + C\frac{\|(\alpha_0, \beta_0)\|_{B^1_{\infty,1}\cap L^2}}{\|\beta_0\|_{B^1_{\infty,1}}}\right)\right\}.$$

However, the regime of small β is not of great interest, since in that case the MHD system (4.5) degenerates. For this reason, we focus on the result of Theorem 4.10.

[6] The importance of solving the ideal MHD system in spaces based on L^p for $p \neq 2$ is plain here: for solutions (α, β) with critical regularity $s = 1 + d/p$, the vorticities have regularity $s = d/p$, which is zero only when $p = +\infty$.

4.5.3 Proof of Theorem 4.10

We now present the proof of Theorem 4.10, which we divide into three main steps. First, we will seek $B^1_{\infty,1}$ estimates on the solution that make the magnetic field appear. Secondly, as the magnetic field equation involves higher order derivatives of the solution, we will find $B^2_{\infty,1}$ bounds in order to close our first estimates. After this, we will only have to perform a few computations to close the proof.

Step 1: an estimate with loss of derivatives.

Although the data possess additional regularity, the various continuation criteria of the previous Chapter (for example Corollary 3.23) show that it is enough to bound the lifespan in the lower regularity space $B^1_{\infty,1}$. For this, we define

$$\mathcal{E}(t) := \big\|(\alpha_0,\beta_0)\|_{L^2} + \big\|\big(X(t),Y(t)\big)\big\|_{B^0_{\infty,1}} \approx \big\|\big(\alpha(t),\beta(t)\big)\|_{L^2\cap B^1_{\infty,1}}\,.$$

Let us focus for a while on bounding the Besov norm of the vorticities X and Y. Since they solve system (4.3), linear estimates for the transport equation (Theorem 3.18 in [7]) yield

$$\begin{aligned}\big\|\big(X(t),Y(t)\big)\big\|_{B^0_{\infty,1}}&\\ &\lesssim \left(1+\int_0^t \|(\nabla\alpha,\nabla\beta)\|_{L^\infty}\,\mathrm{d}\tau\right)\left\{\big\|\big(X_0,Y_0\big)\big\|_{B^0_{\infty,1}} + \int_0^t \|\mathcal{L}(\nabla\alpha,\nabla\beta)\|_{B^0_{\infty,1}}\mathrm{d}\tau\right\}.\end{aligned}$$

The key point of our proof is that, owing to (4.24) above, the integral term in the second line is in fact linear with respect to ∇b. By use of Lemma 4.5 and relation (4.24), we gather

$$\begin{aligned}\big\|\mathcal{L}(\nabla\alpha,\nabla\beta)\big\|_{B^0_{\infty,1}} = 2\,\big\|\mathcal{L}(\nabla u,\nabla b)\big\|_{B^0_{\infty,1}} &\lesssim \|\nabla u\|_{L^\infty}\,\|b\|_{B^1_{\infty,1}} + \|\nabla b\|_{L^\infty}\,\|u\|_{B^1_{\infty,1}}\\ &\lesssim \|b\|_{B^1_{\infty,1}}\,\mathcal{E}(t)\,.\end{aligned}$$

Those inequalities and the conservation of energy $\|(\alpha,\beta)\|_{L^2} = \|(\alpha_0,\beta_0)\|_{L^2}$ give a differential inequality for $\mathcal{E}(t)$,

$$\mathcal{E}(t) \lesssim \left(1+\int_0^t \mathcal{E}(\tau)\,\mathrm{d}\tau\right)\left\{\mathcal{E}(0) + \int_0^t \mathcal{E}(\tau)\,\|b\|_{B^1_{\infty,1}}\,\mathrm{d}\tau\right\}. \tag{4.25}$$

In order to use this inequality, we must find bounds on the magnetic field which depend only on the initial value b_0. We therefore resort to the magnetic field equation, to which we apply the dyadic block Δ_j. We get

$$\big(\partial_t + u\cdot\nabla\big)\Delta_j b = \Delta_j\big((b\cdot\nabla)u\big) + \big[u\cdot\nabla,\Delta_j\big]b\,.$$

By using the tame estimates (1.12) and commutator Lemma 3.14, we may estimate the right-hand side in L^∞ by

$$2^j\,\big\|\Delta_j\big((b\cdot\nabla)u\big)\big\|_{L^\infty} + 2^j\,\big\|\big[u\cdot\nabla,\Delta_j\big]b\big\|_{L^\infty} \lesssim c_j(t)\,\|b\|_{B^1_{\infty,1}}\,\|u\|_{B^2_{\infty,1}}\,,$$

where, as usual, the sequence $\big(c_j(t)\big)_j$ belongs to ℓ^1, with $\sum_{j\geq -1} c_j(t) = 1$. This gives rise to a differential inequality

$$\big\|b(t)\big\|_{B^1_{\infty,1}} \lesssim \big\|b_0\big\|_{B^1_{\infty,1}} + \int_0^t \|b(\tau)\|_{B^1_{\infty,1}}\,\|u(\tau)\|_{B^2_{\infty,1}}\,\mathrm{d}\tau\,,$$

on which an application of Grönwall's lemma yields a bound for the magnetic field depending on the initial value b_0 and higher order quantities:

$$\big\|b(t)\big\|_{B^1_{\infty,1}} \lesssim \big\|b_0\big\|_{B^1_{\infty,1}}\,\exp\left(C\int_0^t \|u(\tau)\|_{B^2_{\infty,1}}\,\mathrm{d}\tau\right). \tag{4.26}$$

Inserting (4.26) into inequality (4.25) gives

$$\mathcal{E}(t) \lesssim \left(1 + \int_0 \mathcal{E}(\tau)\,\mathrm{d}\tau\right) \times \left\{\mathcal{E}(0) + \|b_0\|_{B^1_{\infty,1}} \int_0^t \mathcal{E}(\tau) \exp\left(\int_0^\tau \big\|\big(\alpha(s),\beta(s)\big)\big\|_{B^2_{\infty,1}}\,\mathrm{d}s\right)\mathrm{d}\tau\right\}. \tag{4.27}$$

Step 2: bounding the higher order norms.

Inequality (4.27) presents an apparent one derivative loss. The key point, now, is to find a way to write $B^2_{\infty,1}$ bounds for (α,β) by using only the function $\mathcal{E}(t)$. For simplicity of notation, let us introduce the quantity

$$\mathcal{H}(t) := \big\|\big(\alpha(t),\beta(t)\big)\big\|_{L^2\cap B^2_{\infty,1}}$$

and notice that, by virtue of (4.16), we have

$$\mathcal{H}(t) \approx \big\|\big(\alpha(t),\beta(t)\big)\big\|_{L^2} + \big\|\big(X(t),Y(t)\big)\big\|_{B^1_{\infty,1}}. \tag{4.28}$$

The function $\mathcal{H}(t)$ is the higher regularity quantity which we wish to estimate by $\mathcal{E}(t)$ in order to close the bounds of Step 1. For this, we will have to take full advantage of the bounds obtained in Lemma 4.5 for the bilinear term $\mathcal{L}$ and in commutator Lemma 4.6, quite similarly to the way we proved the various continuation criteria in the previous Chapter. These Lemmata allow us to replace (4.9) by the better inequality

$$\big\|\big(X(t),Y(t)\big)\big\|_{B^1_{\infty,1}} \lesssim \mathcal{H}(0) + \int_0^t \Big(\|\nabla\alpha\|_{L^\infty} + \|\nabla\beta\|_{L^\infty}\Big)\mathcal{H}(\tau)\mathrm{d}\tau.$$

where we have used also (4.28) to control the initial data and the $B^2_{\infty,1}$ norm of (α,β) appearing inside the integral in (4.9). From this inequality and (4.28) again, we deduce another inequality

$$\mathcal{H}(t) \lesssim \mathcal{H}(0) + \int_0^t \mathcal{H}(\tau)\mathcal{E}(\tau)\mathrm{d}\tau$$

to which Grönwall's lemma may be applied, finally leading to the upper bound

$$\mathcal{H}(t) \lesssim \mathcal{H}(0)\,\exp\left\{C\int_0^t \mathcal{E}(\tau)\mathrm{d}\tau\right\}. \tag{4.29}$$

Step 3: end of the proof.

All that remains to do is to use inequalities (4.27) and (4.29) to find bounds on $\mathcal{E}(t)$ on a good time interval. With that idea in mind, define the time T^* by

$$T^* := \sup\left\{T > 0 \,\Big|\, \|b_0\|_{B^1_{\infty,1}} \int_0^T \mathcal{E}(t)\exp\left(\int_0^t \mathcal{H}\right)\mathrm{d}t \leq \mathcal{E}(0)\right\},$$

so that inequality (4.27) gives

$$\mathcal{E}(T) \lesssim \mathcal{E}(0)\left(1 + \int_0^T \mathcal{E}\right).$$

We may apply Grönwall's lemma to the above and obtain a the estimate $\mathcal{E}(T) \lesssim \mathcal{E}(0)\exp(cT\mathcal{E}(0))$ which holds for all times $T\in[0,T^*[$. We will make use of this inequality in (4.29) in order to find a lower bound for T^*. To simplify notations in the next computations, we introduce the functions

$$f(t) := c\mathcal{H}(0)e^{c\mathcal{H}(0)t} \qquad \text{and} \qquad F(t) := \int_0^t f(\tau)\,\mathrm{d}\tau = e^{c\mathcal{H}(0)t} - 1.$$

We need to bound from above the integral appearing in the definition of T^*. By substituting our newfound bound for $\mathcal{E}(T)$ and noting that $\mathcal{E}(0) \leq \mathcal{H}(0)$, we get

$$\begin{aligned}\int_0^T \mathcal{E}(t) \exp\left(\int_0^t \mathcal{H}\right) \mathrm{d}t &\lesssim \int_0^T \mathcal{H}(0) e^{ct\mathcal{H}(0)} \exp\left\{\int_0^t \mathcal{H}(0) \exp\left[c\int_0^\tau \mathcal{H}(0) e^{cs\mathcal{H}(0)} \mathrm{d}s\right] \mathrm{d}\tau\right\} \mathrm{d}t \\ &= \int_0^T f(t) \exp\left\{\int_0^t \mathcal{H}(0) e^{cF(\tau)} \mathrm{d}\tau\right\} \mathrm{d}t\end{aligned}$$

In order to bound the integral in the exponential, we note that $e^{c\mathcal{H}(0)t} \geq 1$, so that we can introduce a $f(t)$ term in the integral while preserving the inequalities:

$$\begin{aligned}\int_0^T \mathcal{E}(t) \exp\left(\int_0^t \mathcal{H}\right) \mathrm{d}t &\lesssim \int_0^T f(t) \exp\left\{\int_0^t \mathcal{H}(0) e^{c\mathcal{H}(0)\tau} e^{cF(\tau)} \mathrm{d}\tau\right\} \mathrm{d}t \\ &= \int_0^T f(t) \exp\left\{\int_0^t f(t) e^{cF(\tau)} \mathrm{d}\tau\right\} \mathrm{d}t \\ &= \int_0^T f(t) \exp\left\{\left(e^{cF(t)} - 1\right) \mathrm{d}\tau\right\} \mathrm{d}t\end{aligned}$$

Finally, performing the same trick of letting exponentials appear while preserving the inequalities, we obtain the upper bound, which holds for all $T \leq T^*$,

$$\int_0^T \mathcal{E}(t) \exp\left(\int_0^t \mathcal{H}\right) \mathrm{d}t \lesssim \exp\left\{\exp\left[\exp\left(cT\mathcal{H}(0)\right) - 1\right] - 1\right\} - 1\,,$$

By using this in the definition of T^*, we see that the time T^* can be no smaller than

$$T^* \geq \frac{c}{\mathcal{H}(0)} \log\left\{1 + \log\left[1 + \log\left(1 + \frac{\mathcal{E}(0)}{\|b_0\|_{B^1_{\infty,1}}}\right)\right]\right\}$$

which proves the Theorem.

4.6 Additional Comments on Plane MHD and Linear Solutions

In this final Section, we explore more deeply the algebraic structure of the plane MHD equations. We will start be providing further comments ont the bilinear term $\mathcal{L}$ which appears in the vorticity form of ideal MHD. Next, we will study the symmetric gradient $[\nabla_\sigma f]_{i,j} = \frac{1}{2}(\partial_i f_j + \partial_j f_i)$ of the ideal MHD system, and we will find the set of equations solved by $\nabla_\sigma \alpha$ and $\nabla_\sigma \beta$. In a third and last paragraph, we will use this structure to study exact solutions of the 2D ideal MHD system that are *linear* with respect to the space variable, and show that they are global in time.

The few computations displayed below have not been previously published. Nevertheless, we include them in the dissertation in the hope that they might help understanding the specific features of plane MHD.

4.6.1 Further Insight on the Bilinear Term

The bilinear term $\mathcal{L}(\nabla\alpha, \nabla\beta)$ appearing in the vorticity form of the Elsässer system can be seen as the main obstacle to a theory of global solutions to the 2D ideal MHD system: it is precisely its absence in the Euler problem which provides all known global well-posedness results, while its presence in our equations (4.3) is a formidable challenge, given that it is a critically scaled[7] and non-local function of (X, Y).

[7] In other words, $\nabla\alpha$ and $\nabla\beta$ are images of the quantities X and Y by a Fourier multiplier of degree zero.

It could be hoped that a better understanding of the way $\mathcal{L}$ contributes to the evolution of X and Y would help to determine whether solutions of the 2D ideal MHD system can be global or might blow-up in finite time. With that in mind, we will compute the exact value of $\mathcal{L}(\nabla\alpha,\nabla\beta)$ on a family of known exact solutions (see Subsection 2.2.1 in [81] about these). Consider, for smooth functions $\phi,\psi:\mathbb{R}_+\longrightarrow\mathbb{R}$, the stationary flow defined in polar[8] coordinates (r,θ)

$$u=\phi(r)e_\theta \qquad \text{and} \qquad b=\psi(r)e_\theta,$$

where (e_r,e_θ) is the basis of $\mathbb{R}^2$ adapted to polar coordinates $e_r=(\cos(\theta),\sin(\theta))$ and $e_\theta=(-\sin(\theta),\cos(\theta))=e_r^\perp$. First of all, it is easy to check that u and b are both divergence-free, as they are the orthogonal gradients of radial functions

$$u=\nabla^\perp\left[\int_0^r\phi\right]:=\nabla^\perp\Phi(r) \qquad \text{and} \qquad b=\nabla^\perp\left[\int_0^r\psi\right]:=\nabla^\perp\Psi(r).$$

Next, we see that (u,b) are indeed solutions of the ideal MHD equations. On the one hand, projecting the 3D magnetic field equation $\partial_t b=\nabla\times(u\times b)$ on the plane gives

$$\partial_t b-\nabla^\perp(u\cdot b^\perp)=\partial_t b=0,$$

because u and b are indeed parallel. On the other hand, taking the curl of the momentum equation and using the fact that both u and b have no divergence gives

$$\partial_t\omega+u\cdot\nabla\omega-b\cdot\nabla j=\partial_t\omega=0$$

because both $\omega=\Delta\Phi$ and $j=\Delta\Psi$ are radial functions, and so must have gradients that are orthogonal to e_θ.

Remark 4.12. By taking both ϕ and ψ compactly supported, we construct an infinite family of exact solutions of the plane ideal MHD system that are confined to a bounded area. It should be noted that this possibility is a specific property of two dimensions $d=2$. A virial type argument due to Shafranov [98] (see also paragraph 9.8.2, pp. 276-278 in [11]) shows that ther can be no static magnetofluid configuration confined in a bounded region of space if $d=3$. The computation rests on the rate of decay of magnetic dipole fields and does not work when $d=2$.

Now that we have made sure that (u,b) is indeed a solution of the ideal MHD equations, computing $\mathcal{L}(\nabla u,\nabla b)$ is not difficult. Because of the form of our solution, β and ∇X are orthogonal (remember $X=\omega+j$ is radial), so that we must have

$$\mathcal{L}(\nabla\alpha,\nabla\beta)=\partial_t X+\beta\cdot\nabla X=0.$$

On the other hand, we may also compute $\mathcal{L}(\nabla u,\nabla b)$ in a more direct fashion in order to highlight the structure of $\mathcal{L}$. Let us first write ∇u and ∇b explicitly. Recall that the gradient operator ∇ is the transpose of the differential $[\nabla f]_{i,j}=\partial_i f_j$, so that

$$\nabla u=\phi'(r)\,e_r\otimes e_\theta-\frac{1}{r}\phi(r)\,e_\theta\otimes e_r,$$

and likewise for the magnetic field. In order to perform our computations, we will project ∇u and ∇b on the following basis on the space of traceless 2×2 matrices

$$E_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad E_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad E_3=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \tag{4.30}$$

[8] From now on, we agree that (r,θ) is the function of x implicitly defined by $x=(x_1,x_2)=r(\cos(\theta),\sin(\theta))$. The gradient operator ∇ is to be computed with respect to x.

which are orthogonal for the so-called Frobenius scalar product of matrices $\langle A, B\rangle = \mathrm{Tr}({}^tAB)$. By expressing the tensor products $e_r \otimes e_\theta$ and $e_\theta \otimes e_r$ in the canonical basis, we obtain

$$\nabla u = -\cos(\theta)\sin(\theta)\left(\phi'(r) - \frac{1}{r}\phi(r)\right)E_1 \\ + \frac{1}{2}\big(\cos^2(\theta) - \sin^2(\theta)\big)\left(\phi'(r) - \frac{1}{r}\phi(r)\right)E_2 + \frac{1}{2}\left(\phi'(r) + \frac{1}{r}\phi(r)\right)E_3$$

and likewise for ∇b. Because of (4.23), the quantity $\mathcal{L}(\nabla u, \nabla b)$ can be seen to depend only on the E_1 and E_2 components of ∇u and ∇b. In slightly different terms, we may note that $\mathcal{L}$ can be written as a determinant

$$\mathcal{L}(\nabla\alpha, \nabla\beta) = -\begin{vmatrix} \partial_1\alpha_2 & \partial_1\beta_2 & 1 \\ \partial_2\alpha_1 & \partial_2\beta_1 & -1 \\ \partial_1\alpha_1 & \partial_1\beta_1 & 0 \end{vmatrix}$$

which can be interpreted as a determinant in the 3D space of traceless matrices involving ∇u, ∇b and E_3. Either way, the terms depending on ϕ and ψ may be factored out of $\mathcal{L}$ and we have

$$\mathcal{L}(\nabla u, \nabla b) = \left(\phi'(r) - \frac{1}{r}\phi(r)\right)\left(\psi'(r) - \frac{1}{r}\psi(r)\right)\mathcal{L}\big(F(\theta), F(\theta)\big) = 0\,,$$

where $F(\theta)$ is the function defined by

$$F(\theta) := -\cos(\theta)\sin(\theta)E_1 + \frac{1}{2}\big(\cos^2(\theta) - \sin^2(\theta)\big)E_2.$$

Though the usefulness of the computation above is not tremendous, we may still gather a few lessons from it. Firstly, as expected, for a "nice" solution the bilinear term $\mathcal{L}$ is equally "nice". This leaves us with the question of how $\mathcal{L}$ may behave near a blow-up solution and whether it might contribute to avert said blow-up or precipitate it. Secondly, we might note that if $\phi(r) = r$ (or $u(x) = x^\perp$), then $r\phi'(r) = \phi(r)$ so that $\mathcal{L}(\nabla u, \nabla f) = 0$ for all divergence-free f. Considering a velocity field whose flow simply rotates the plane of the fluid cancels $\mathcal{L}$ *whatever the form of the magnetic field*. If vice versa $b = x^\perp$, then the bilinear term also cancels regardless of the form of the velocity field. Lastly, we have seen that $\mathcal{L}(\nabla\alpha, \nabla\beta)$ depends only on the E_1 and E_2 components of $\nabla\alpha$ and $\nabla\beta$, that is on the elements of the symmetric gradients

$$[\nabla_\sigma u]_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \qquad \text{and} \qquad [\nabla_\sigma b]_{ij} = \frac{1}{2}\left(\partial_i b_j + \partial_j b_i\right).$$

It could therefore be of interest to find what kind of evolution equations $\nabla_\sigma u$ and $\nabla_\sigma b$ solve. This is the purpose of the next paragraph.

4.6.2 Symmetric Gradient Equations

In this Subsection, we seek to determine what PDE system the symmetric gradients $\nabla_\sigma u$ and $\nabla_\sigma b$ solve. We will use the calculations below in the next paragraph.

By taking the symmetric gradient of the Elsässer system (4.5), we find that $\nabla_\sigma(\alpha, \beta)$ are solutions of transport-like equations:

$$\big(\partial_t + \beta\cdot\nabla\big)[\nabla_\sigma\alpha]_{ij} + \partial_i\partial_j\pi + \frac{1}{2}\Big(\partial_i\beta_k\partial_k\alpha_j + \partial_j\beta_k\partial_k\alpha_i\Big) = 0. \tag{4.31}$$

We will organize the non-transport part of this equation in order to make the different components of the derivatives $\nabla\alpha$ and $\nabla\beta$ appear. More precisely, we will be interested in the components with respect to the basis (E_1, E_2, E_3) of the space of traceless 2×2 matrices defined in (4.30) above. We introduce the following notation: for every function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, set

$$\nabla f := \partial_1 f_1 E_1 + D_\sigma f E_2 + \frac{1}{2}\mathrm{curl}\,(f) E_3, \tag{4.32}$$

so that $\mathrm{curl}\,(f) = \partial_1 f_2 - \partial_2 f_1$ as usual and $D_\sigma f = \frac{1}{2}(\partial_1 f_2 + \partial_2 f_1)$. With all this in mind, our first step is to write (4.31) as a sum of traceless matrices. Define

$$F_{ij} := \frac{1}{2}\Big(\partial_i \beta_k \partial_k \alpha_j + \partial_j \beta_k \partial_k \alpha_i\Big)\,,$$

so that the $(1,1)$ component of F is, by using the fact that α and β are divergence free,

$$\begin{aligned}
F_{11} &= \partial_1\alpha_1\partial_1\beta_1 + \partial_1\beta_2\partial_2\alpha_1 \\
&= \frac{1}{2}\Big(\partial_1\alpha_1\partial_1\beta_1 + \partial_2\alpha_2\partial_2\beta_2 + \partial_1\beta_2\partial_2\alpha_1 + \partial_2\beta_1\partial_1\alpha_2\Big) + \frac{1}{2}\Big(\partial_1\beta_2\partial_2\alpha_1 - \partial_2\beta_1\partial_1\alpha_2\Big) \\
&= -\frac{1}{2}\Delta\pi + \frac{1}{2}\Big(\partial_1\beta_2\partial_2\alpha_1 - \partial_2\beta_1\partial_1\alpha_2\Big).
\end{aligned}$$

The last line is obtained by remembering that the MHD pressure solves the elliptic equation $-\Delta\pi = \partial_j\partial_k(\alpha_j\beta_k)$ obtained by taking the divergence of the equations. Let us express the last summand in terms of the components defined in (4.32) by noting that the cross derivatives $\partial_1 f_2$ and $\partial_2 f_1$ are given by $D_\sigma f \pm \frac{1}{2}\mathrm{curl}\,(f)$. We therefore obtain

$$F_{11} = -\frac{1}{2}\Delta\pi + \frac{1}{2}\Big(Y D_\sigma \alpha - X D_\sigma \beta\Big),$$

while the expression for F_{22} can be recovered simply from reversing the roles of α and β in the previous equation. Concerning the cross components $F_{12} = F_{21}$, we can make the vorticities appear by writing

$$\begin{aligned}
F_{12} = F_{21} &= \frac{1}{2}\Big(\partial_1\beta_1\partial_1\alpha_2 + \partial_1\beta_2\partial_2\alpha_2 + \partial_2\beta_1\partial_1\alpha_1 + \partial_2\beta_2\partial_2\alpha_1\Big) \\
&= \frac{1}{2}\partial_1\beta_1 X - \frac{1}{2}\partial_1\alpha_1 Y.
\end{aligned}$$

Putting the expressions for the coordinates F_{ij} together in equation (4.31), we obtain

$$\Big(\partial_t + \beta\cdot\nabla\Big)\nabla_\sigma\alpha + \nabla^2\pi - \frac{1}{2}\Delta\pi.\mathrm{Id} + \frac{1}{2}\Big(YD_\sigma\alpha - XD_\sigma\beta\Big)E_1 + \frac{1}{2}\Big(\partial_1\beta_1 X - \partial_1\alpha_1 Y\Big)E_2 = 0. \tag{4.33}$$

The equation for $\nabla_\sigma\beta$ can be obtaining by reversing the roles of α and β in (4.33). As a final step, we slightly reformulate the last two summands in the equation by introducing the rotation matrix

$$R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that putting all this into (4.33) gives a system of equation for $\nabla_\sigma\alpha$ and $\nabla_\sigma\beta$

$$\begin{cases} \big(\partial_t + \beta\cdot\nabla\big)\nabla_\sigma\alpha + \nabla^2\pi - \frac{1}{2}\Delta\pi.\mathrm{Id} = \frac{1}{2}\big(YR\nabla_\sigma\alpha - XR\nabla_\sigma\beta\big) \\ \big(\partial_t + \alpha\cdot\nabla\big)\nabla_\sigma\beta + \nabla^2\pi - \frac{1}{2}\Delta\pi.\mathrm{Id} = \frac{1}{2}\big(XR\nabla_\sigma\beta - YR\nabla_\sigma\alpha\big). \end{cases} \tag{4.34}$$

Remark 4.13. A certain symmetry should be noted in (4.33) and in the vorticity equations (4.3). On the one hand, by (orthogonally) projecting (4.33) on E_1 and E_2, we see that the equations for the E_1 component $\partial_a\alpha_a$ involves only the curls and D_σ elements, and the equation on the E_2 component $D_\sigma\alpha$ involves only $\partial_1\beta_1$, $\partial_1\alpha_1$ and the curls X and Y. This must be put in parallel with the vorticity equations (the E_3 components) whose righthand side term involves only the elements of $\nabla_\sigma(\alpha,\beta)$, as

$$\mathcal{L}(\nabla\alpha,\nabla\beta) = \partial_1\alpha_1 D_\sigma\beta - \partial_1\beta_1 D_\sigma\alpha.$$

Any equation on one E_k component of $\nabla(\alpha,\beta)$ involves a skew-symmetric expression of the two other components.

Remark 4.14. The presence of the second derivatives of the pressure may seem inappropriate for the obtention of energy estimates. It turns out that it is not the case, as integration by parts provides

$$\int \big(\partial_i\partial_j\pi - \frac{1}{2}\delta_{i,j}\partial_k^2\pi\big)\big(\partial_i\alpha_j + \partial_j\alpha_i\big)\,\mathrm{d}x = \int \Delta\pi\,\mathrm{div}(\alpha)\,\mathrm{d}x = 0$$

Next, there is an equivalence of norms between $\|\nabla_\sigma\alpha\|_{L^2}$ and $\dot{H}^1$. In fact, divergence-free functions achieve the equality in the Korn inequality: for all (not necessarily divergence-free) $f:\mathbb{R}^d\longrightarrow\mathbb{R}^d$, integration by parts yield

$$\int |\nabla_\sigma f|^2\,\mathrm{d}x = \frac{1}{2}\int |\nabla f|^2\,\mathrm{d}x + \frac{1}{2}\int \mathrm{div}(f)^2\,\mathrm{d}x,$$

so that $\|f\|_{\dot{H}^1} \leq \sqrt{2}\|\nabla_\sigma f\|_{L^2}$. It is therefore possible to perform energy estimates in system (4.34). By taking the matrix (Frobenius) scalar product (see the explanation immediately after (4.30)) of the first equation in (4.34), we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla_\sigma\alpha|^2\,\mathrm{d}x &= \frac{1}{2}\int Y\mathrm{Tr}(\nabla_\sigma\alpha R\nabla_\sigma\alpha)\,\mathrm{d}x - \frac{1}{2}\int X\mathrm{Tr}(\nabla_\sigma\alpha R\nabla_\sigma\beta)\,\mathrm{d}x\\ &= -\frac{1}{2}\int X\mathrm{Tr}(\nabla_\sigma\alpha R\nabla_\sigma\beta)\,\mathrm{d}x\\ &= \frac{1}{4}\int X\mathcal{L}(\nabla\alpha,\nabla\beta)\,\mathrm{d}x,\end{aligned}$$

so that the energy estimates obtained from the vorticity equations (4.3) by testing them against X and Y can be recovered in this way. Incidentally, note that we have

$$\mathcal{L}(\nabla\alpha,\nabla\beta) = -\frac{1}{2}\mathrm{Tr}(\nabla_\sigma\alpha R\nabla_\sigma\beta). \tag{4.35}$$

4.6.3 Linear Solutions of 2D MHD

In this paragraph, we follow the ideas of Miller [84], who proposed the study of linear solutions (see Proposition 1.5 in [81]) of the Navier-Stokes equations and proved their finite time blowup. We consider solutions of the plane ideal MHD equations (4.5) of the form

$$\alpha(t,x) = A(t).x \qquad \text{and} \qquad \beta(t,x) = B(t).x, \tag{4.36}$$

where $A,B\in C^1(\mathbb{R}^2\otimes\mathbb{R}^2)$ are time dependent matrices. The goal will be to find differential equations satisfied by A and B and see whether these may induce finite time blowup or not. As notes the author of [84] (see Remark 2.8), the philosophy of this toy model is quite different from the usual approach: that is replacing a difficult equation by a relaxed problem which can be studied in the key function spaces. Instead, we study exact solutions of (4.5) but in a very comfortable functional setting.

Reformulation as a Differential System. The first step will be to find the system of differential equations satisfied by $A(t)$ and $B(t)$. We will write the equations in all space dimensions $d \geq 2$, although our study will be limited to the plane case $d = 2$. In order to substitute (4.36) in the MHD equations, we first note that, since $\nabla\alpha = {}^tA$,

$$\big[(\beta \cdot \nabla)\alpha\big]_i = B_{kl}x_l\partial_k(A_{ij}x_j) = B_{kl}A_{ij}\delta_{kj}x_l = A_{ik}B_{kl}x_l = \big[AB.x\big]_i$$

The divergence-free condition for these solutions is simply written $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 0$. By defining the pressure $\pi = \pi_1 = \pi_2$ by

$$\pi = -\frac{1}{2d}\mathrm{Tr}(AB)|x|^2, \tag{4.37}$$

so that $\nabla\pi = -\frac{1}{d}\mathrm{Tr}(AB)x$, the triplet (α, β, π) defines a solution of the ideal MHD system on the condition that

$$A'.x + AB.x - \frac{1}{d}\mathrm{Tr}(AB)x = 0 \qquad \text{and} \qquad \mathrm{Tr}(A) = 0,$$

with a symmetric equation for $B.x$. We summarize the computations above with the following Definition.

Definition 4.15. Let $T > 0$ and $d \geq 2$. Consider a set of initial data $(A_0, B_0) \in \mathbb{R}^d \otimes \mathbb{R}^d$ such that $\mathrm{Tr}(A_0) = \mathrm{Tr}(B_0) = 0$. The functions $(A, B) \in C^1([0, T[; \mathbb{R}^d \otimes \mathbb{R}^d)$ are said to be *linear solutions of the ideal MHD equations* if they solve the differential system

$$\begin{cases} A' + AB - \frac{1}{d}\mathrm{Tr}(AB)\mathrm{Id} = 0 \\ B' + BA - \frac{1}{d}\mathrm{Tr}(BA)\mathrm{Id} = 0 \end{cases} \tag{4.38}$$

with initial data (A_0, B_0). Note that the solution is automatically traceless at all positive times $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 0$.

Remark 4.16. The author of [84] notes that it is possible to choose the pressure (4.37) in multiple ways while still defining exact solutions of the PDE problem (see Remark 2.11) for the same initial datum. In particular, given a set (A_0, B_0) of initial data, there is an infinity of exact solutions of the ideal MHD system (4.5) of the form (4.36) corresponding to different choices of the pressure. This must be put in relation with our results of Chapter 2 where we have shown that this is a typical property of incompressible fluids when solutions have no decay properties at $|x| \to +\infty$. However, assuming that the pressure is given by (4.37), there is a unique solution of the ideal MHD system of the form (4.36).

Solving the Equations. From now on, we work in two dimensions $d = 2$. In order to solve equations (4.38) above, we will resort to the vorticity and symmetric gradient equations (4.3) and (4.38). This corresponds to taking the skew-symmetric and symmetric parts of the matrix equations (4.38). We introduce the following notation:

$$A_\sigma = \frac{1}{2}(A + {}^tA) \qquad \text{and} \qquad X = A_{21} - A_{12},$$

and likewise for B, so that we do in fact have $\nabla_\sigma\alpha = A_\sigma$ and $\mathrm{curl}\,(\alpha) = X$ provided α is given by (4.36). Then importing this in systems (4.3) and (4.38) yields, with the help of (4.35), the differential system[9]

$$\begin{cases} A'_\sigma = \frac{1}{2}\big(YRA_\sigma - XRB_\sigma\big) \\ B'_\sigma = \frac{1}{2}\big(XRB_\sigma - YRA_\sigma\big) \\ X' = -\frac{1}{2}\mathrm{Tr}(A_\sigma RB_\sigma) \\ Y' = -\frac{1}{2}\mathrm{Tr}(B_\sigma RA_\sigma). \end{cases}$$

[9]The quantities X and $\nabla_\sigma\alpha$ solve independent equations. This should not be surprising: because of the special form of the solutions, there is no hope of expressing $\nabla\alpha$ in terms of X or $\nabla_\sigma\alpha$ only. The Biot-Savart law and its analogue $\nabla f = \nabla(-\Delta)^{-1}\,\mathrm{div}(\nabla_\sigma f)$ no longer hold in this framework.

The miracle is that these equations are in fact *linear*, thanks to the skew-symmetry of all righthand sides. Adding the equations for A_σ and B_σ and the equations for X and Y, one finds two constant quantities:

$$(A_\sigma + B_\sigma)' = 0 \qquad \text{and} \qquad (X+Y)' = 0,$$

so that, by setting $C_\sigma := A_\sigma + B_\sigma$ and $C_3 := X + Y$, we find that all the non-linear terms cancel and give

$$\begin{cases} A_\sigma' = \frac{1}{2}\big(C_3 R A_\sigma - X R C_\sigma\big) \\ X' = -\frac{1}{2}\mathrm{Tr}(C_\sigma R A_\sigma), \end{cases}$$

so in particular the solutions are *global in time*. As a matter of fact, we may go one step further and completely solve this differential system. In order to put the previous differential equations in matrix form, we set

$$C_\sigma := \begin{pmatrix} C_1 & C_2 \\ C_2 & -C_1 \end{pmatrix} \qquad \text{and} \qquad A_\sigma := \begin{pmatrix} y_1 & y_2 \\ y_2 & -y_1 \end{pmatrix}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \\ X \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -C_3 & C_2 \\ C_3 & 0 & -C_1 \\ C_2 & -C_1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ X \end{pmatrix}.$$

Let M be the 3×3 matrix appearing in this differential equation. The form of the solutions will be determined by the eigenvalues of M, which are roots of the polynomial

$$\begin{vmatrix} X & C_3 & -C_2 \\ -C_3 & X & C_1 \\ -C_2 & C_1 & X \end{vmatrix} = X(X^2 + C_3^2 - C_1^2 - C_2^2).$$

In other words, there is one conserved quantity (corresponding to the eigenvector $(C_1, C_2, C_3) \in \ker(M)$), and the solutions are either exponentially increasing or periodic depending on the relative size of C_3 and C_σ.

To conclude our discussion, let us explain the meaning of these results in terms of the ideal MHD system.

1. Firstly, $C_3 = \omega$ is the vorticity and $C_\sigma = \nabla_\sigma u$ is the deformation tensor of the flow. Therefore, if $(y_1, y_2, X)_{t=0} = (C_1, C_2, C_3)$, then $\nabla\beta = 0$, and the equations degenerate to $u \equiv b$ (see Remark 4.11).

2. In the case where $b = 0$, so that (4.36) is a solution of the 2D Euler system, then $\alpha = \beta$ and $(y_1, y_2, X) = \frac{1}{2}(C_1, C_2, C_3)$. Once more the equations degenerate to

   $$A_\sigma' = B_\sigma' = 0 \qquad \text{and} \qquad X' = Y' = 0,$$

   which is in fact the matrix form of the 2D Euler system.

3. The growth of (non-degenerate) solutions is decided by the sign of the quantity

   $$C_3^2 - C_1^2 - C_3^2 = \omega^2 - |\nabla_\sigma u|^2.$$

   Therefore, if the vorticity is larger than the deformation tensor, then the solutions are periodic. If not, they are exponentially increasing. This principle might be linked to the fact that potential blowup of usual solutions can be seen on the velocity field alone (see Theorem 3.25). One may also wonder if the relative size of ω and $\nabla_\sigma u$ will have anything to do with the growth of solutions, at least in special regimes of slow evolving fluids. But this is even more hypothetical, as the properties we exhibit in this paragraph might only be a consequence of the special cancellations specific to the framework of linear solutions.

Chapter 5

Fast Rotation Asymptotics

All exact science is dominated by the idea of approximation.
Bertrand Russell[1]

5.1 Introduction

In this Chapter, we study the $\epsilon \to 0^+$ asymptotics of the following non-homogeneous incompressible MHD system with Coriolis force:

$$
(5.1) \qquad \begin{cases} \partial_t \rho + \operatorname{div}\big(\rho\, u\big) \,=\, 0 \\ \partial_t \big(\rho u\big) + \operatorname{div}\big(\rho\, u \otimes u\big) + \dfrac{1}{\epsilon}\,\nabla \Pi + \dfrac{1}{\epsilon}\,\rho\, u^{\perp} \,=\, h(\epsilon) \operatorname{div}\big(\nu(\rho)\,\nabla u\big) + \operatorname{div}\big(b \otimes b\big) - \dfrac{1}{2}\nabla |b|^2 \\ \partial_t b + \operatorname{div}\big(u \otimes b\big) - \operatorname{div}\big(b \otimes u\big) \,=\, h(\epsilon) \nabla^{\perp}\big(\mu(\rho)\operatorname{curl}(b)\big) \\ \operatorname{div}(u) \,=\, 0\,. \end{cases}
$$

These equations are set in a two dimensional domain Ω, which is either the plane $\mathbb{R}^2$ or the torus $\mathbb{T}^2$. The vector fields u and b are the velocity and the magnetic fields, the scalar fields $\rho \geq 0$ and Π represent the density and the pressure fields respectively. The two functions of the density $h(\epsilon)\nu(\rho)$ and $h(\epsilon)\mu(\rho)$ represent the viscosity and resistivity coefficients respectively, which we allow to depend on the rotation parameter $\epsilon > 0$. Notice that there is no specific reason for considering the viscous stress tensor of the form $\nu(\rho)\,\nabla u$, and other choices (for instance, replacing ∇u with the symmetric gradient $D(u) = \big(\nabla u + {}^t\nabla u\big)/2$) can be accommodated with minor changes in the analysis.[2] The notation $u^{\perp}$ refers the rotation of angle $\pi/2$ of the vector u: if $u = (u_1, u_2)$, then $u^{\perp} = (-u_2, u_1)$. Analogously, we have set $\nabla^{\perp} = (-\partial_2, \partial_1)$. We have also defined $\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$ to be the curl of the 2-D vector u. As we will see, since the first equation is written up to a gradient field, the term $\nabla |b|^2/2$ does not appear in the weak form of the equations.

The main goal of this chapter is to study the asymptotics of the solutions $(\rho_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$ of system (5.1), and proving that they converge (in some way) to solutions of a limit evolution PDE system.

5.1.1 General Physical Remarks

We focus on fluids on which the Coriolis force $\epsilon^{-1}\rho u^{\perp}$ has a major influence compared to the kinematics of the said fluid, such as large-scale fluids evolving on a celestial body. The importance

[1] This quote is attributed to Russell in B. Winckler, *op. cit.*

[2] Other choices are possible, with deeper mathematical consequences, and will not be considered here. For instance, alternate models include anisotropic scaling in the viscosity to take into account the joint effects of turbulence and the asymmetry induced by the rotation, see [21] and [36] for more on this topic.

of this effect is measured by the Rossby number of the fluid $Ro = \epsilon$, the condition $\epsilon \ll 1$ defining the regime of large-scale planetary or stellar fluid dynamics. At the mathematical level, in the limit $\epsilon \to 0^+$, the Coriolis force can only be balanced by the pressure term, which is reflected by the ϵ^{-1} factor in front of $\nabla\Pi$.

We are not lacking examples for which this is a relevant model. The geophysical approximation is well suited to oceanic or atmospheric flows, as well as terrestrial magma (as in the study of the dynamo effect). In addition, the assumption of a Coriolis force that dominates the dynamics is not limited to *geo*physical fluids, as it is equally a helpful tool for investigating stellar fluids or extraterrestrial weather. For instance, the rotation speed of Jupiter is so high that the study of the Jovian atmosphere could be considered a textbook case.

Physical Relevance of the System

This paragraph is devoted to a few critical remarks concerning system (5.1), and the physical setting that led to its derivation.

First of all, the model neglects any effect due to temperature variations, which is a debatable simplification, even in the case of non-conducting fluids. For instance, ocean water density is an intricate function of the pressure, salinity and temperature, and the temperature of air masses plays a major role in weather evolution. In those cases, dependence on the pressure is often neglected, and both temperature and salinity are assumed to evolve through a diffusion process (see [31], Chapter 3). For conducting fluids, which are generally heated magma or plasmas, the temperature is expected to play an even greater role. However, the equations, as they are, already provide interesting challenges and are widely used by physicists for practical purposes.

Secondly, let us give a few comments concerning the two-dimensional setting. Our main motivation for restricting to 2-D domains is purely technical: in our analysis, we face similar difficulties as in [46], devoted to the fast rotation asymptotics for density-dependent incompressible Navier-Stokes equations in dimension $d = 2$. In particular, the fast rotation limit for incompressible non-homogeneous fluids in 3-D is a widely open problem (see more details here below). However, let us notice that one of the common features of highly rotating fluids is to be, in a first approximation, planar: the fluid is devoid of vertical motion and the particles move in columns. This property is known as the Taylor-Proudman theorem (see [31], [88] for useful insight). Therefore, the 2-D setting is in itself a relevant approximation for geophysical fluids.

At this point, note that equations (5.1) *per se* do not describe a conducting fluid confined to a quasi-planar domain. If that were the case, the magnetic field would circulate *around* the current lines, hence being orthogonal to the plane of the fluid, assuming the form $b = b_3(t,x)e^3$ for some scalar function b_3. Our problem, which involves a 2-D magnetic field $b = (b_1, b_2)$ is a projection of the full three-dimensional MHD system, and is therefore linked to a different physical setting: a solution (ρ, u, b, Π) of the two dimensional MHD equations can be extended into a solution (ρ', u', b', Π') of the 3D problem by setting

$$\rho'(t,x) = \rho(t,x_1,x_2), \qquad \Pi'(t,x) = \Pi(t,x_1,x_2)$$

and

$$u'(t,x) = \big(u(t,x_1,x_2),0\big), \qquad b'(t,x) = \big(b(t,x_1,x_2),0\big)$$

for $x = (x_1,x_2,x_3) \in \mathbb{R}^3$, thus defining a 3D solution invariant under translation along the x_3 axis. Taking this step away from the physical problem brings us closer to the actual form of the physically relevant 3-D problem, and we hope it will provide a step towards its understanding.

Next, we remark that we have taken a quite simple form for the Coriolis force: namely, $\mathfrak{C}[\rho,u] = \rho u^\perp$. This means that the rotation axis is constant and normal to the plane where the fluid moves. Of course, more complicated choices are possible. However, on the one hand this form for the rotation term is physically consistent with a fluid evolving at mid-latitudes, in a region small enough compared to the radius of the planetary or stellar body. On the other hand, this choice is quite common in mathematical studies, and the obtained model is already able to explain several physical phenomena.

Finally, we point out that the incompressibility assumption $\mathrm{div}(u) = 0$ is a valid approximation for flows in the ocean and in the atmosphere, and we will assume it. On the other hand, by our choice of considering domains with a very simple geometry, we completely avoid boundary effects.

5.1.2 Goal of our Study

We aim at understanding the limit dynamics of (5.1) when $\epsilon \to 0^+$ by showing that the solutions $(\rho_\epsilon, u_\epsilon, b_\epsilon)$ converge in some precise way to solutions of a limit system of evolution PDEs. In this process, grasping exactly how the fluid density behaves is absolutely crucial, as in the regime of fast rotations, we see from the form of the Coriolis force

$$F_C = \frac{1}{\epsilon}\rho u^\perp = \frac{1}{Ro}\rho u^\perp$$

that any non-homogeneity will be amplified by the Coriolis force. We therefore expect that even slight perturbations in the fluid density will have an impact on the dynamics. Let us explain how this happens. The only other force that is able to compensate this effect is, at the geophysical scale, the pressure force, which must therefore scale as $1/\epsilon$,

$$\frac{1}{\epsilon}\rho U^\perp \approx \frac{1}{\epsilon}\nabla\Pi.$$

Assume now that the fluid is quasi-homogeneous, so that density is nearly constant $\rho = 1 + \epsilon R$. Then, the Coriolis force reads

$$F_C = \frac{1}{\epsilon}U^\perp + RU^\perp.$$

Because the fluid is incompressible, the dominant term $\epsilon^{-1}U^\perp$ in this equation is in fact the gradient of some function: it has no curl. Therefore, it does not appear in the weak form of the incompressible MHD equations, and the Coriolis force only ends up contributing $RU^\perp$ to the dynamics.

With this in mind, we may take the formal limit of equations (5.1) under the assumption that the density is quasi-homogeneous $\rho = 1 + \epsilon R$. We obtain the following *quasi-homogeneous MHD system*

$$\begin{cases} \partial_t U + \mathrm{div}(U \otimes U - B \otimes B) + RU^\perp + \nabla\left(\Pi + \frac{1}{2}|B|^2\right) = h(0)\nu(1)\Delta U \\ \partial_t B + \mathrm{div}(U \otimes B - B \otimes U) = h(0)\mu(1)\Delta B \\ \partial_t R + \mathrm{div}\left(RU\right) = 0 \\ \mathrm{div}(U) = \mathrm{div}\,(B) = 0. \end{cases} \tag{5.2}$$

From this rough discussion, we expect some kind of convergence theorem, from solutions of (5.1) to those of (5.2). There will be two distinct cases : the first where the viscosity and resistivity remain non-degenerate $h(\epsilon) = 1$, and a second where we consider evanescent dissipation $h(\epsilon) \to 0^+$. The first case has already received our attention in [27] with a method based on compensated compactness. In that case, there is weak convergence

$$\left(\frac{1}{\epsilon}(1-\rho_\epsilon), u_\epsilon, b_\epsilon\right) \quad \rightharpoonup \quad (R, U, B), \qquad \text{for } \epsilon \to 0^+, \tag{5.3}$$

of the solutions of (5.1) to those (5.2) with $h(0) = 1$. However, since the convergence (5.3) follows from compactness methods and is not necessarily in a norm topology, there is no explicit speed of convergence. In addition, as the method used in [27] uses bounds for the solutions of (5.1) in the energy space $L^\infty(L^2) \cap L^2(\dot{H}^1)$, the case of vanishing viscosity and resistivity $h(\epsilon) \to 0^+$ is not handled.

In this Chapter, we use a relative entropy inequality (see [51] for more on this method) for problem (5.1) to prove quantitative convergence results of the solutions of (5.1) in both cases: to those of (5.2) when $h(\epsilon) \equiv 1$, and those of (5.2) when $h(\epsilon) \to 0^+$. What this amounts to is a structure theorem: solutions of (5.1) are the sum of a solution of (5.2) (or (5.2)) plus a remainder whose L^2 norm has limit zero as $\epsilon \to 0^+$.

5.1.3 Previous Mathematical Results on Fast Rotating Fluids

The mathematical study of rotating fluids is by no means new in the mathematical literature. It has started in the 1990s with the pioneering works [4]-[5]-[6] of Babin, Mahalov and Nikolaenko, and has since been deeply investigated, above all for models of homogeneous incompressible fluids. We refer to book [21] for a complete treatment of the incompressible Navier-Stokes equations with Coriolis force, and for further references on this subject.

The study of fast rotation asymptotics for non-homogeneous fluids has a much more recent history. However, efforts have mainly been focused on compressible fluid models: see e.g. [50], [49], [53], [52], [44], [74]. We refer to [45] for additional details and further references, as well as for recent developments. On the contrary, not as many results are available for density-dependent incompressible fluids. To the best of our knowledge, the only work in this direction prior to our own is [46], treating the case of the non-homogeneous Navier-Stokes equations in two-dimensional domains. The reason for such a gap between the compressible and the (non-homogeneous) incompressible cases is that the coupling between the mass and the momentum equation is weaker in the latter situation than in the former one. As a consequence, less information is available on the limit points of the sequences of solutions, and taking the limit in the equations becomes a harder task. This explains also the lack of results for 3-D incompressible flows with variable densities.

The case of rotating MHD equations has also received some attention in the past years. Once again, most of the available results concern the case of homogeneous flows: for instance, we mention papers [36] and [92], concerning the stability of boundary layers in homogenous rotating MHD, and [86], about the stabilizing effect the rotation has on solution lifespan. See also references therein, as well as Chapter 10 of [21], for further references. On the density-dependent side, fast rotating asymptotics has recently been conducted in [73] for compressible flows, in two space dimensions. We point out that the approach of [73] is based on relative entropy estimates; if on the one hand this method enables to consider also a vanishing viscosity and resistivity regime, on the other hand it requires to assume well prepared initial data.

The results we present in this Chapter come from our paper [28]. They build on our previous work [27], where we had analyzed the fast-rotation limit of (5.1) for *ill-prepared* initial data and shown weak convergence of the solutions (as in (5.3) above). The difference between the two papers [27] and [28] is twofold.

The former article [27] deals with general target densities $\rho_\epsilon \rightharpoonup \rho_0(x)$, and not only $\rho_0(x) = 1$ as above. This introduces a complication with respect to the dissipation coefficients, as convergence of the coefficients

$$\nu(\rho_\epsilon) \longrightarrow \nu(\rho_0) \qquad \text{and} \qquad \mu(\rho_\epsilon) \longrightarrow \mu(\rho_0)$$

is highly non-trivial, due to the fact that convergence of the density may not be strong. Compare with the assumption above that the density is quasi-homogeneous $\rho_\epsilon = 1 + \epsilon R_\epsilon$, immediately providing the pointwise convergence needed to deal with the dissipation coefficients, whereas in [27] we had to resort to Di Perna-Lions theory.

Secondly, the weak framework of [27] makes it harder to take the limit in the other non-linear quantities, and especially the convective[3] term $\operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon)$. The answer to this was a compensated compactness argument similar to that of [46].

On the other hand, because the results of [27] imply only weak convergence of the solutions of (5.1) by compactness arguments, even in the case of a homogeneous target density, see (5.3), it is not possible to recover an explicit convergence speed with this method. The purpose of the Chapter (and our paper [28]) is to provide such a rate of convergence by means of a relative entropy inequality. This consists roughly of taking the difference between (5.1) and the target system (5.2), so that having (unique) solutions of the limit equations is paramount to the proof. In the case where $h(0) = 1$, this is mainly solving a Navier-Stokes type system (see [27]), while in the vanishing dissipation case $h(0) = 0$, we need to resort to the theory of solutions developed in the previous Chapters of this dissertation (see also [28]).

Notation and conventions

Before starting, let us introduce some useful notation that are specific to this Chapter and which will be used throughout all our study.

- The space domain will be denoted by $\Omega \subset \mathbb{R}^d$: throughout the text, we will always work in the case $d = 2$.
- Let $\big(f_\epsilon\big)_{\epsilon>0}$ be a sequence of functions in a normed space X. If this sequence is bounded in X, we use the notation $\big(f_\epsilon\big)_{\epsilon>0} \subset X$.
- In all the text, $M_p(t) \in L^p(\mathbb{R}_+)$ will be a generic globally L^p function; on the other hand, we will use the notation $N_p(t)$ to denote a generic function in $L^p_{\mathrm{loc}}(\mathbb{R}_+)$.

5.2 The Primitive System

In this Section, we define the class of solutions of the primitive and target systems with which we will work : on the one hand, we will always deal with finite energy solutions of (5.1), while the class of solutions of the limit equations (5.2) will depend on whether we are in the parabolic case $h(0) = 1$ or the ideal $h(0) = 0$ one.

First of all, the viscosity coefficient ν and the resistivity coefficient μ are assumed to be continuous and non-degenerate: more precisely, they satisfy

$$\nu\,,\ \mu \in C^0(\mathbb{R}_+)\,, \qquad \text{with, } \forall\, \rho \geq 0\,, \quad \nu(\rho) \geq \nu_* > 0 \quad \text{and} \quad \mu(\rho) \geq \mu_* > 0\,, \tag{5.4}$$

for some positive real numbers ν_* and μ_*. The function h satisfies either $h \equiv 1$ (see Theorem 5.4) or $h \in C^0([0,1])$, with $h \geq 0$ and $h(\epsilon) \longrightarrow 0$ for $\epsilon \to 0^+$ (see Theorem 5.7).

For any $\varepsilon > 0$, we consider initial data $\big(\rho_{0,\varepsilon}, u_{0,\varepsilon}, b_{0,\varepsilon}\big)$ to system (5.1), and a corresponding global in time finite energy weak solution $\big(\rho_\varepsilon, u_\varepsilon, b_\varepsilon\big)$, in the sense specified by Definition 5.1 below.

5.2.1 Initial Data, Finite Energy Weak Solutions

We supplement system (5.1) with general ill prepared initial data. Let us be more precise, and start by considering the density functions: for any $0 < \varepsilon \leq 1$, we take

$$\rho_{0,\epsilon} \,=\, 1 \,+\, \epsilon\, r_{0,\epsilon}\,, \qquad \text{with} \qquad \big(r_{0,\epsilon}\big)_{\epsilon>0} \,\subset\, \big(L^2 \cap L^\infty\big)(\Omega)\,.$$

[3]The othe quadratic terms involving the magnetic field turn out to be simpler to handle, especially since they to no contain the density.

Without loss of generality we can suppose that there exist two constants $0 < \rho_* \leq \rho^*$ such that, for all $\varepsilon > 0$, one has

$$0 \, < \, \rho_* \, \leq \, \rho_{0,\epsilon} \, \leq \, \rho^* \, . \tag{5.5}$$

Therefore, by the first equation in (5.1), vacuum can never appear. For this reason, we can work directly with the velocity fields $u_{0,\varepsilon}$, which is therefore always well defined, unlike in vacuum bubbles where the velocity of the fluid particules has no precise meaning. We assume u to be such that

$$\big(u_{0,\epsilon}\big)_{\epsilon>0} \, \subset \, L^2(\Omega) \qquad \text{and} \qquad \mathrm{div}\,(u_{0,\epsilon}) \, = \, 0 \, .$$

Finally, for the magnetic fields, we choose initial data

$$\big(b_{0,\epsilon}\big)_{\epsilon>0} \, \subset \, L^2(\Omega) \qquad \text{and} \qquad \mathrm{div}\,(b_{0,\epsilon}) \, = \, 0 \, .$$

Before going on, we remark that the assumption $\rho_{0,\epsilon} \, = \, 1 \, + \, \epsilon \, r_{0,\epsilon}$ simplifies the equations very much. Indeed, since $\mathrm{div}\, u_\varepsilon = 0$ for any $\varepsilon > 0$, at any later time we still have $\rho_\epsilon \, = \, 1 \, + \, \epsilon \, r_\epsilon$, with r_ϵ solving a linear transport equation

$$\partial_t r_\epsilon + \mathrm{div}(r_\epsilon u_\epsilon) = 0 \, , \qquad\qquad \big(r_\epsilon\big)_{|t=0} = r_{0,\epsilon} \, .$$

We are now ready to introduce the definition of finite energy weak solution to system (5.1). Here below, the notation $C_w\big([0,T];L^2(\Omega)\big)$ stands for the space of time-dependent functions, taking values in L^2, which are continuous with respect to the weak topology of that space.

Definition 5.1. Let $T > 0$ and $\epsilon \in]0,1]$ fixed. Let $\big(\rho_{0,\epsilon}, u_{0,\epsilon}, b_{0,\epsilon}\big)$ be an initial datum fulfilling the previous assumptions. We say that $\big(\rho, u, b\big)$ is a *finite energy weak solution* to system (5.1) in $[0,T] \times \Omega$, related to the previous initial datum, if the following conditions are verified:

(i) $\rho \in L^\infty\big([0,T] \times \Omega\big)$ and $\rho \in C\big([0,T]; L^q_{\mathrm{loc}}(\Omega)\big)$ for all $1 \leq q < +\infty$;

(ii) $u \in L^\infty\big([0,T]; L^2(\Omega)\big) \cap C_w\big([0,T]; L^2(\Omega)\big)$, with $\nabla u \in L^2\big([0,T]; L^2(\Omega)\big)$;

(iii) $b \in L^\infty\big([0,T]; L^2(\Omega)\big) \cap C_w\big([0,T]; L^2(\Omega)\big)$, with $\nabla b \in L^2\big([0,T]; L^2(\Omega)\big)$;

(iv) the mass equation is satisfied in the weak sense: for any $\phi \in \mathcal{D}\big([0,T] \times \Omega\big)$, one has

$$\int_0^T \int_\Omega \Big\{\rho \partial_t \phi + \rho u \cdot \nabla \phi\Big\} \mathrm{d}x\mathrm{d}t + \int_\Omega \rho_{0,\epsilon}\phi(0,\cdot)\mathrm{d}x = \int_\Omega \rho(T)\phi(T,\cdot)\,\mathrm{d}x \, ;$$

(v) the divergence-free conditions $\mathrm{div}(u) = \mathrm{div}(b) = 0$ are satisfied in $\mathcal{D}'\big(\,]0,T[\,\times \Omega\big)$;

(vi) the momentum and magnetic field equations are satisfied in the weak sense: for any $\psi \in \mathcal{D}\big([0,T] \times \Omega; \mathbb{R}^2\big)$ such that $\mathrm{div}(\psi) = 0$, one has

$$\int_0^T \int_\Omega \Big\{\rho u \cdot \partial_t \psi + \big(\rho u \otimes u - b \otimes b\big) : \nabla \psi - \frac{1}{\epsilon} \rho u^\perp \cdot \psi - h(\epsilon)\nu(\rho)\nabla u : \nabla \psi\Big\} \mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega \rho_{0,\epsilon} u_{0,\epsilon} \cdot \psi(0,\cdot)\mathrm{d}x = \int_\Omega \rho(T)u(T) \cdot \psi(T,\cdot)\,\mathrm{d}x \, ,$$

and for all $\zeta \in \mathcal{D}\big([0,T] \times \Omega; \mathbb{R}^2\big)$ one has

$$\int_0^T \int_\Omega \Big\{b \cdot \partial_t \zeta + \big(u \otimes b - b \otimes u\big) : \nabla \zeta + h(\epsilon)\mu(\rho)\big(\nabla \times b\big)\big(\nabla \times \zeta\big)\Big\} \mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega b_{0,\epsilon} \cdot \zeta(0,\cdot)\,\mathrm{d}x = \int_\Omega b(T) \cdot \zeta(T,\cdot)\,\mathrm{d}x \, ;$$

(vii) for almost every $t \in [0,T]$, the following energy balance holds true:

$$\int_\Omega \Big(\rho(t)|u(t)|^2 + |b(t)|^2\Big)\,\mathrm{d}x + \int_0^t \int_\Omega h(\epsilon)\Big(\nu(\rho)|\nabla u|^2 + \mu(\rho)|\nabla \times b|^2\Big)\,\mathrm{d}x\mathrm{d}\tau$$
$$\leq \int_\Omega \Big(\rho_{0,\epsilon}|u_{0,\epsilon}|^2 + |b_{0,\epsilon}|^2\Big)\,\mathrm{d}x\,.$$

The solution $\big(\rho, u, b\big)$ is said to be *global* if the above conditions hold for all $T > 0$.

Before going on, we still have a few definitions and assumptions to give. First of all, owing to the previous uniform bounds for the initial data, we deduce that, up to an extraction, one has the weak convergence properties

$$(5.6)\quad u_{0,\epsilon} \rightharpoonup U_0 \quad \text{in } L^2(\Omega)\,, \qquad r_{0,\epsilon} \overset{*}{\rightharpoonup} R_0 \quad \text{in } \big(L^2 \cap L^\infty\big)(\Omega)\,, \qquad b_{0,\epsilon} \rightharpoonup B_0 \quad \text{in } L^2(\Omega)\,,$$

for suitable functions U_0, R_0 and B_0 belonging to the respective functional spaces. Notice that we obviously have the strong convergence property $\rho_{0,\epsilon} - \rho_0 \longrightarrow 0$ in $L^2 \cap L^\infty$ for $\epsilon \to 0^+$.

In order to to derive quantitative estimates for solutions to (5.1), we need more precise assumptions on the viscosity and resistivity coefficients than the ones in (5.4) above. We start by a definition (see e.g. Section 2.10 of [7] for details).

Definition 5.2. A *modulus of continuity* is a continuous non-decreasing function $\sigma : [0,1] \longrightarrow \mathbb{R}_+$ such that $\sigma(0) = 0$.

Given a modulus of continuity σ, the space $\mathcal{C}_\sigma(\mathbb{R})$ is defined as the set of real-valued functions $a \in L^\infty(\mathbb{R})$ such that

$$|a|_{\mathcal{C}_\mu} := \sup_{x\in\mathbb{R}} \sup_{|y|\in]0,1]} \frac{|a(x+y) - a(x)|}{\sigma(|y|)} < +\infty\,.$$

We also define $\|a\|_{\mathcal{C}_\sigma} := \|a\|_{L^\infty} + |a|_{\mathcal{C}_\sigma}$.

In view of the previous definition, beside (5.4), we also assume that there exists a modulus of continuity σ such that

$$\nu\,,\ \mu \in \mathcal{C}_\sigma(\mathbb{R})\,.$$

Strictly speaking, we only need this in a neighborhood of $\rho = 1$, but we formulate the previous global assumption for simplicity of presentation.

5.2.2 The Viscous and Resistive Case

Under the hypotheses formulated in the previous section, and for the choice $h \equiv 1$ in (5.1), it was shown in [27] that the limit dynamics for $\epsilon \to 0^+$ is described by the quasi-homogeneous MHD system

$$(5.7)\qquad \begin{cases} \partial_t R + \mathrm{div}(RU) = 0 \\ \partial_t U + \mathrm{div}(U \otimes U) + \nabla\left(\Pi + \dfrac{|B|^2}{2}\right) + RU^\perp = \nu(1)\,\Delta U + \mathrm{div}(B \otimes B) \\ \partial_t B + \mathrm{div}(U \otimes B - B \otimes U) = \mu(1)\,\Delta B \\ \mathrm{div}(U) = \mathrm{div}(B) = 0\,, \end{cases}$$

for some suitable pressure function π and with the triplet $\big(R_0, U_0, B_0\big)$, identified in (5.6) above, as initial datum. In addition, we have shown that equations (5.7) are well posed (this is Theorem 5.1 in [27]). More precisely, the following theorem holds true.

Theorem 5.3. *Consider* $0<\beta<1$ *and* $(R_0,U_0,B_0)\in H^{1+\beta}(\Omega)\times H^1(\Omega)\times H^1(\Omega)$.

For that initial datum, there is exactly one solution (R,U,B) *of* (5.7) *in the energy space, that is such that* $R\in L^\infty\big(\mathbb{R}_+;(L^2\cap L^\infty)(\Omega)\big)$ *and* $U,B\in L^\infty\big(\mathbb{R}_+;L^2(\Omega)\big)$, *with* $\nabla U,\nabla B\in L^2\big(\mathbb{R}_+;H^1(\Omega)\big)$. *Moreover, this unique solution satisfies* $R\in L^\infty_{\rm loc}\big(\mathbb{R}_+;H^{1+\gamma}(\Omega)\big)$ *for all* $\gamma<\beta$, *and* $U,B\in L^\infty_{\rm loc}\big(\mathbb{R}_+;H^1(\Omega)\big)\cap L^2_{\rm loc}\big(\mathbb{R}_+;H^2(\Omega)\big)$.

With this theorem at hand, we are ready to state the quantitative convergence result in the viscous and resistive case.

Theorem 5.4. *Let* $h\equiv 1$ *in* (5.1) *and* ν, μ *be as in* (5.4). *For a given modulus of continuity* σ, *assume in addition that* $\nu,\mu\in C_\sigma(\mathbb{R})$. *Consider a sequence* $\big(\rho_{0,\epsilon},u_{0,\epsilon},b_{0,\epsilon}\big)_{\epsilon>0}$ *of initial data satisfying the assumptions fixed in Subsection 5.2.1, and let* $\big(\rho_\epsilon,u_\epsilon,b_\epsilon\big)_{\epsilon>0}$ *be a corresponding sequence of global in time finite energy weak solutions to system* (5.1). *Define* $M>0$ *by*

$$M\;:=\;\sup_{\varepsilon>0}\|r_{0,\epsilon}\|_{L^\infty}\;+\;\sup_{\varepsilon>0}\|u_{0,\epsilon}\|_{L^2}\;+\;\sup_{\varepsilon>0}\|b_{0,\epsilon}\|_{L^2}\,.$$

Assume also that the triplet $\big(R_0,U_0,B_0\big)$, *defined in* (5.6), *belongs to* $H^{1+\beta}(\Omega)\times H^1(\Omega)\times H^1(\Omega)$, *for some* $\beta\in\,]0,1[\,$, *and let* $\big(R,U,B\big)$ *be the corresponding unique solution to system* (5.7), *as given by Theorem 5.3. Finally, set*

$$\delta r_\varepsilon\;:=\;r_\varepsilon-R\,,\qquad \delta u_\varepsilon\;:=\;u_\varepsilon-U\,,\qquad \delta b_\varepsilon\;:=\;b_\varepsilon-B$$

and, with analogous notation, $\delta r_{0,\varepsilon}\;:=\;r_{0,\varepsilon}-R_0$, $\delta u_{0,\varepsilon}\;:=\;u_{0,\varepsilon}-U_0$ *and* $\delta b_{0,\varepsilon}\;:=\;b_{0,\varepsilon}-B_0$.

Then, for all fixed times $T>0$, *the following estimate holds true: for any* $\varepsilon>0$ *and almost every* $t\in[0,T]$,

$$\begin{aligned}(5.8)\quad \|\delta r_\epsilon(t)\|_{L^2}^2+\|\delta u_\epsilon(t)\|_{L^2}^2+\|\delta b_\epsilon(t)\|_{L^2}^2+\int_0^t\Big\{\|\nabla\delta u_\epsilon\|_{L^2}^2+\|\nabla\times\delta b_\epsilon\|_{L^2}^2\Big\}\,{\rm d}\tau&\\ \leq C\bigg\{\|\delta r_{0,\epsilon}\|_{L^2}^2+\|\delta u_{0,\epsilon}\|_{L^2}^2+\|\delta b_{0,\epsilon}\|_{L^2}^2+\max\big\{\epsilon^2\,,\,\sigma^2(M\epsilon)\big\}\bigg\}&\,,\end{aligned}$$

where the constant $C>0$ *depends on* T, *on the lower bounds* ν_* *and* μ_* *as well as on* $|\nu|_{\mathcal{C}_\sigma}$ *and* $|\mu|_{\mathcal{C}_\sigma}$, *on the norms of the initial data* $\|u_0\|_{H^1}$, $\|b_0\|_{H^1}$ *and* $\|r_0\|_{H^{1+\beta}}$, *and on* M.

Remark 5.5. In the simpler case when $\mu,\nu\in C^1(\mathbb{R}_+)$, the last summand in the brackets (of the right-hand side of (5.8)) becomes $O(\epsilon^2)$. Note that the convergence does not improve when e.g. μ and ν are constant near $\rho=1$. As we will see below, this is mainly due to the fact that $\rho_\epsilon=1+O(\epsilon)$.

The previous theorem immediately yields the following corollary, which is in fact a convergence result: based on the strong convergence of the initial data, we deduce strong convergence of the solutions of (5.1) to the solution of (5.7). This is very much in the spirit of [73] and [52].

Corollary 5.6. *Under the same assumptions as in Theorem 5.4 above, assume moreover that*

$$\|r_{0,\epsilon}-r_0\|_{L^2}^2+\|u_{0,\epsilon}-u_0\|_{L^2}^2+\|b_{0,\epsilon}-b_0\|_{L^2}^2\underset{\epsilon\to0^+}{\longrightarrow}0\,.$$

Then, for all fixed $T>0$, *we have strong convergence of the solutions*

$$\big(u_\epsilon,b_\epsilon\big)\underset{\epsilon\to0^+}{\longrightarrow}\big(U,B\big)\quad\textit{in}\;\;L^\infty_T(L^2)\cap L^2_T(H^1)\qquad\textit{and}\qquad r_\epsilon\underset{\epsilon\to0^+}{\longrightarrow}R\quad\textit{in}\;\;L^\infty_T(L^2)\,.$$

5.2.3 Vanishing Viscosity and Resistivity Limit: the Ideal Case

In this paragraph, we consider the vanishing viscosity and resistivity limit of system (5.1). Namely, we now assume that

$$(5.9) \qquad h \in C^0([0,1])\,, \qquad \forall\, \epsilon > 0\,, h(\epsilon) > 0 \qquad \text{and} \qquad h(\epsilon) \underset{\epsilon \to 0^+}{\longrightarrow} 0\,.$$

The main challenge this new problem poses is that, because the elliptic parts of the equations vanish in (5.1), we can no longer rely on *uniform* $L^2_{\mathrm{loc}}(H^1)$ bounds for the velocity and the magnetic fields. We must therefore require additional regularity on the limit points, which solve the quasi-homogeneous ideal MHD system (analogous to (5.2) above)

$$(5.10) \qquad \begin{cases} \partial_t R + \mathrm{div}(RU) = 0 \\ \partial_t U + \mathrm{div}(U \otimes U - B \otimes B) + \nabla\pi + RU^\perp = 0 \\ \partial_t B + \mathrm{div}(U \otimes B - B \otimes U) = 0 \\ \mathrm{div}(U) = \mathrm{div}(B) = 0. \end{cases}$$

The quasi-homogeneous ideal MHD system above is very close to the usual ideal MHD system which has been at the center of our attention in the first four Chapters of the dissertation. Its well-posedness is not much harder, mainly due to the fact that the velocity is coupled with the density perturbation function only through a order zero rotation term $RU^\perp$. In [28] and [29] we have proved local existence and uniqueness of solutions for (5.10) in Besov-Lipschitz spaces with pretty much the same methods as Chapters 3 and 4. For simplicity, we are going to work on energy spaces H^s, corresponding to the choice $r = p = 2$. We point out that Theorem 5.7 below holds on any time interval $[0, T]$ on which solutions of the limit problem exist.

Theorem 5.7. *Assume that assumptions* (5.4) *and* (5.9) *hold for the coefficients* ν, μ *and* h *in system* (5.1). *Consider a sequence* $\big(\rho_{0,\epsilon}, u_{0,\epsilon}, b_{0,\epsilon}\big)_{\epsilon>0}$ *of initial data satisfying the assumptions fixed in Subsection 5.2.1, and let* $\big(\rho_\epsilon, u_\epsilon, b_\epsilon\big)_{\epsilon>0}$ *be a corresponding sequence of global in time finite energy weak solutions to system* (5.1). *Define* $M > 0$ *as above, i.e.*

$$M \,:=\, \sup_{\varepsilon>0} \|r_{0,\epsilon}\|_{L^\infty} \,+\, \sup_{\varepsilon>0} \|u_{0,\epsilon}\|_{L^2} \,+\, \sup_{\varepsilon>0} \|b_{0,\epsilon}\|_{L^2}\,.$$

Assume also that the triplet $\big(R_0, U_0, B_0\big)$, *defined in* (5.6), *belongs to* $(H^s(\Omega))^3$, *for some* $s > 2$, *and let* $\big(R, U, B\big) \in C^0\big([0, T^*[\,; H^s(\Omega)\big)^3$ *be the corresponding unique strong solution to system* (5.10), *where* $T^* > 0$ *denotes the maximal time of existence of that solution. Finally, set*

$$\delta r_\varepsilon \,:=\, r_\varepsilon - R\,, \qquad \delta u_\varepsilon \,:=\, u_\varepsilon - U\,, \qquad \delta b_\varepsilon \,:=\, b_\varepsilon - B$$

and, with analogous notation, $\delta r_{0,\varepsilon} \,:=\, r_{0,\varepsilon} - R_0$, $\delta u_{0,\varepsilon} \,:=\, u_{0,\varepsilon} - U_0$ *and* $\delta b_{0,\varepsilon} \,:=\, b_{0,\varepsilon} - B_0$.

Then, for all fixed times $T \in [0, T^*[$, *the following estimate holds true: for any* $\varepsilon > 0$ *and almost every* $t \in [0, T]$,

$$(5.11) \quad \|\delta r_\epsilon(t)\|^2_{L^2} + \|\delta u_\epsilon(t)\|^2_{L^2} + \|\delta b_\epsilon(t)\|^2_{L^2} + h(\epsilon) \int_0^t \Big\{ \|\nabla \delta u_\epsilon\|^2_{L^2} + \|\nabla \times \delta b_\epsilon\|^2_{L^2} \Big\}\, \mathrm{d}\tau$$
$$\leq C \Big\{ \|\delta r_{0,\epsilon}\|^2_{L^2} + \|\delta u_{0,\epsilon}\|^2_{L^2} + \|\delta b_{0,\epsilon}\|^2_{L^2} + \epsilon^2 + h(\epsilon) \Big\}\,,$$

where the constant $C > 0$ *depends on the lower bounds* ν_* *and* μ_*, *as well as on the norm of the initial datum* $\|(R_0, U_0, B_0)\|_{H^s}$, *on* M *and on* T.

Of course, a corollary in the spirit of Corollary 5.6 can also be deduced in this case.

5.3 A Relative Entropy Inequality for the Primitive System

In this section we show that any finite energy weak solution to the non-homogeneous viscous and resistive MHD system (5.12) satisfies a relative entropy inequality.

For simplicity, we will assume the space domain Ω to be either the whole space $\mathbb{R}^2$ or the torus $\mathbb{T}^2$. However, more general domains may be allowed at this stage.

5.3.1 Preliminaries

Let us collect here our main working assumptions, which will be assumed to hold true throughout the rest of this section. They simply correspond to taking $\epsilon = 1$ in Subsections 5.2 and 5.2.1 above; however, for reader's convenience, we will recall them here.

To begin with, the primitive system (recall (5.1) and take $\epsilon = 1$) is now the following non-homogeneneous incompressible MHD system with Coriolis force:

$$
(5.12) \qquad \begin{cases} \partial_t \rho + \operatorname{div}\big(\rho u\big) = 0 \\ \partial_t(\rho u) + \operatorname{div}\big(\rho\, u \otimes u\big) + \nabla \pi + \rho u^\perp = \operatorname{div}\big(\nu(\rho)\, \nabla u\big) + \operatorname{div}\big(b \otimes b\big) - \dfrac{1}{2} \nabla |b|^2 \\ \partial_t b + \operatorname{div}\big(u \otimes b\big) - \operatorname{div}\big(b \otimes u\big) = \nabla^\perp\big(\mu(\rho)\, \nabla \times b\big) \\ \operatorname{div}(u) = 0\,. \end{cases}
$$

The viscosity and resistivity coefficients, ν and μ respectively, satisfy assumption (5.4): specifically,

$$
\nu\,,\ \mu \in C^0(\mathbb{R}_+)\,, \qquad \text{with, } \forall\, \rho \geq 0\,, \quad \nu(\rho) \geq \nu_* > 0 \quad \text{and} \quad \mu(\rho) \geq \mu_* > 0\,.
$$

System (5.12) is supplemented with the initial datum $\big(\rho\,,\, u\,,\, b\big)_{|t=0} = \big(\rho_0\,,\, u_0\,,\, b_0\big)$ satisfying the following assumptions (see also [58], [78]):

(a) for the initial density, we require

$$
\rho_0 \geq \rho_* > 0 \qquad \text{and} \qquad \rho_0 - 1 := r_0 \in L^2(\Omega) \cap L^\infty(\Omega)\,;
$$

(b) for the initial velocity and magnetic fields, we require

$$
u_0 \in L^2(\Omega)\,, \quad b_0 \in L^2(\Omega) \qquad \text{and} \qquad \operatorname{div} u_0 = \operatorname{div} b_0 = 0\,.
$$

Notice that the L^2 condition on r_0 is not really needed for the existence of weak solutions, but we assume it for later use in Subsection 5.3.2. Similarly, the assumption $\rho_0 \geq \rho_* > 0$ is enough in view of our study of Section 5.4 (recall hypothesis (5.5) above), although it could be slightly relaxed (in the spirit of conditions (2.8) to (2.10) of [78], see also [27]) for formulating the relative entropy inequality. Remark however that the weak solutions theory for system (5.12) requires absence of vacuum (see e.g. [58], [37]).

For a given initial datum (ρ_0, u_0, b_0) verifying the hypotheses above, we consider a global in time finite energy weak solution (r, u, b) of (5.12). Here, the definition of finite energy weak solution is the same as in Definition 5.1 above, with the choice $\epsilon = 1$. In particular, we point out the following facts:

(i) the weak formulation of the momentum equation becomes: for any $\psi \in \mathcal{D}\big([0,T] \times \Omega; \mathbb{R}^2\big)$ such that $\operatorname{div}(\psi) = 0$, one has

$$
(5.13) \qquad \int_0^T \int_\Omega \Big\{\rho u \cdot \partial_t \psi + \big(\rho u \otimes u - b \otimes b\big) : \nabla \psi - \rho u^\perp \cdot \psi - \nu(\rho) \nabla u : \nabla \psi\Big\}\, \mathrm{d}x\mathrm{d}t \\ + \int_\Omega \rho_0 u_0 \cdot \psi(0,\cdot)\mathrm{d}x = \int_\Omega \rho(T) u(T) \cdot \psi(T,\cdot)\, \mathrm{d}x\,;
$$

(ii) the weak formulation of the magnetic field equation now reads: for all $\zeta \in \mathcal{D}\big([0,T] \times \Omega; \mathbb{R}^2\big)$ one has

$$\int_0^T \int_\Omega \Big\{ b \cdot \partial_t \zeta + \big(u \otimes b - b \otimes u\big) : \nabla \zeta + \mu(\rho) \big(\nabla \times b\big) \big(\nabla \times \zeta\big) \Big\} \, \mathrm{d}x\mathrm{d}t \\ + \int_\Omega b_0 \cdot \zeta(0,\cdot)\, \mathrm{d}x = \int_\Omega b(T) \cdot \zeta(T,\cdot)\, \mathrm{d}x\,; \tag{5.14}$$

(iii) the energy inequality becomes: for almost every $t \in [0,T]$, the following energy balance holds true:

$$\int_\Omega \Big(\rho(t)|u(t)|^2 + |b(t)|^2 \Big)\, \mathrm{d}x + \int_0^t \int_\Omega \Big(\nu(\rho)|\nabla u|^2 + \mu(\rho)|\nabla \times b|^2 \Big)\, \mathrm{d}x\mathrm{d}\tau \\ \leq \int_\Omega \Big(\rho_0|u_0|^2 + |b_0|^2 \Big)\, \mathrm{d}x\,. \tag{5.15}$$

Existence of global in time finite energy weak solutions (in the previous sense) to system (5.12) has been shown by Gerbeau and Le Bris in [58] in a bounded domain of $\mathbb{R}^3$; see also [37] for related results and additional references. However, the proof of [58] can be extended to the cases $\mathbb{R}^2$ or $\mathbb{T}^2$ (which are relevant for our study) with standard modifications.

We point out that, in [58], the authors resort to P.-L. Lions's theory [78] for non-homogeneous fluids with density-dependent viscosities but with no magnetic field. In that case, the initial density is even allowed to vanish, under some suitable non-degeneracy condition; on the contrary, the result of [58] holds only for fluids with non-vanishing initial densities, namely $\rho_0 > 0$ (see Remark 3.4 of [58] for more comments about this issue). In view of assumption (5.5), the result of [58] is all what we need for the present study.

Before going on, some remarks are in order.

Remark 5.8. Observe that, owing to the divergence-free condition on the magnetic field and Plancherel's theorem, when the space dimension is $d = 2$ one has

$$\big\|\nabla b\big\|_{L^2} = \big\| - i\xi \widehat{b}(\xi)\big\|_{L^2} = \big\| - i\xi^\perp \cdot \widehat{b}(\xi)\big\|_{L^2} = \big\|\nabla \times b\big\|_{L^2}\,.$$

Therefore, inequality (5.15) above yields an L^2 bound for the full gradient of b.

Remark 5.9. When vacuum is permitted, or even when $\rho_0 > 0$ (without a uniform lower bound), the (weak) time continuity of the velocity field is no longer true in general. On the other hand, one can show that $\mathbb{P}(\rho u)$ is (weakly) continuous in time, where $\mathbb{P}$ is the Leray projector onto the space of divergence-free vector fields (see Theorem 2.2 of [78] and Remark 3.1 of [58] in this respect).

To conclude this part, we point out that, as the fluid density ρ is simply transported by the divergence-free velocity field u, for all $t \geq 0$ we get

$$\forall\, p \in [2,+\infty]\,, \qquad \|\rho(t) - 1\|_{L^p} = \|\rho_0 - 1\|_{L^p}\,, \qquad \|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty}\,. \tag{5.16}$$

5.3.2 Stating the Relative Entropy Inequality

Following what done in Subsection 5.2.1, we set

$$r := \rho - 1 \qquad \text{and} \qquad r_0 := \rho_0 - 1\,, \tag{5.17}$$

and we notice that, owing to the first equation in (5.12) and the condition $\mathrm{div}\,(u) = 0$, one has

$$\partial_t r + \mathrm{div}\,\big(r\,u\big) = 0\,, \qquad r_{|t=0} = r_0\,. \tag{5.18}$$

Since this relation is completely analogous to the mass conservation equation, throughout all this section we will equivalently speak of solutions (ρ, u, b) and (r, u, b) to the MHD system (5.12), implying that r and ρ are linked through (5.17).

Next, we define the *relative entropy* of a solution (r, u, b) of the non-homogeneous system (5.12) with respect to a triplet (R, U, B) of (say) smooth, compactly supported functions in $\mathbb{R}_+ \times \Omega$, to be the quantity

$$\mathcal{E}\Big([r,u,b]\,\Big|\,[R,U,B]\Big) := \frac{1}{2}\int_\Omega \Big\{\rho\,\big|u-U\big|^2 + \big|b-B\big|^2 + \big|r-R\big|^2\Big\}\,\mathrm{d}x\,. \tag{5.19}$$

We can then formulate the following *relative entropy inequality*: for almost every $T>0$,

$$\begin{aligned}\mathcal{E}\Big([r,u,b]\,\Big|\,[R,U,B]\Big)(T) &+ \int_0^T\!\!\int_\Omega \Big(\nu(\rho)\,\big|\nabla(u-U)\big|^2 + \mu(\rho)\,\big|\nabla\times(b-B)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \mathcal{E}\Big([r_0,u_0,b_0]\,\Big|\,[R(0),U(0),B(0)]\Big) + \int_0^T \mathcal{R}\big(r,u,b;R,U,B\big)\,\mathrm{d}t\,,\end{aligned} \tag{5.20}$$

where we have defined

$$\begin{aligned}\mathcal{R}\big(r,u,b;R,U,B\big) := &-\int_\Omega \rho\,\Big(\partial_t U + (u\cdot\nabla)U + U^\perp\Big)\cdot\delta u\,\mathrm{d}x - \int_\Omega \big(\partial_t B + (u\cdot\nabla)B\big)\cdot\delta b\,\mathrm{d}x \\ &-\int_\Omega \big(\partial_t R + u\cdot\nabla R\big)\cdot\delta r\,\mathrm{d}x + \int_\Omega \Big(\big(b\cdot\nabla\big)U\cdot\delta b + \big(b\cdot\nabla\big)B\cdot\delta u\Big)\,\mathrm{d}x \\ &-\int_\Omega \nu(\rho)\,\nabla U:\nabla\delta u\,\mathrm{d}x - \int_\Omega \mu(\rho)\,\big(\nabla\times B\big)\,\big(\nabla\times\delta b\big)\,\mathrm{d}x\,.\end{aligned} \tag{5.21}$$

The main goal of this Section is to prove that any global in time finite energy weak solution to (5.12) satisfies the previous relative entropy inequality, as established in the next statement. Its proof is postponed to the next subsection.

Theorem 5.10. *Let $(r_0, u_0, b_0) \in (L^\infty \cap L^2)(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ be an initial datum for problem* (5.12)*, and let (r, u, b) be a global in time finite energy solution related to that initial datum.*

Then, for any triplet (R, U, B) of functions enjoying the following regularity properties, namely

(1) $R \in W^{1,1}_{\mathrm{loc}}\big(\mathbb{R}_+; L^2(\Omega)\big)$, *with* $\nabla R \in L^2_{\mathrm{loc}}\big(\mathbb{R}_+; L^q(\Omega)\big)$ *for some* $q > 2$,

(2) $U, B \in W^{1,1}_{\mathrm{loc}}\big(\mathbb{R}_+; L^2(\Omega)\big)$, *with* $\nabla U, \nabla B \in L^2_{\mathrm{loc}}\big(\mathbb{R}_+; (L^2 \cap L^q)(\Omega)\big)$ *for some* $q > 2$,

(3) $\mathrm{div}\,(U) = \mathrm{div}\,(B) = 0$ *almost everywhere in* $\mathbb{R}_+ \times \Omega$,

the relative entropy inequality (5.20) *holds for almost every* $T > 0$.

Before going on, let us list a series of possible extensions of the previous result. First of all, Theorem 5.10 can be easily formulated also in the higher dimensional case, provided the Coriolis term vanishes or is changed in an appropriate (physically relevant) way. In addition, the same technique of the proof can be employed to handle more complex geometries of the space domain, encoding different boundary conditions (e.g. no-slip, or complete slip boundary conditions). Finally, as it will appear clearly from the proof, the regularity of the triplet (R, U, B) can also be somehow relaxed, and different integrability hypotheses may be imposed.

However, we refrain from treating such situations in full generality, and we limit ourselves to state and prove the result in the case which is of interest for our applications.

5.3.3 Proof of the Relative Entropy Inequality

In order to prove Theorem 5.10, we proceed as in [51], where the authors prove similar inequalities for compressible Navier-Stokes equations. For simplicity, let us consider for a while a triplet (R, U, B) of smooth functions such that:

(i) $R \in \mathcal{D}\big(\mathbb{R}_+ \times \Omega\big)$;

(ii) U and B belong to $\mathcal{D}\big(\mathbb{R}_+ \times \Omega; \mathbb{R}^2\big)$ and are such that $\operatorname{div} U = \operatorname{div} B = 0$.

Let $T > 0$ be such that the support of R, U and B is included in $[0, T] \times \Omega$. First of all, using $\psi = U$ as a test function in the weak form (5.13) of the momentum equation, and after noting that $u^\perp \cdot U = -u \cdot U^\perp$, we find

(5.22)
$$\int_\Omega \rho(T) u(T) \cdot U(T)\, \mathrm{d}x = \int_\Omega \rho_0 u_0 \cdot U(0)\, \mathrm{d}x + \int_0^T\!\!\int_\Omega \rho u \cdot \partial_t U\, \mathrm{d}x\, \mathrm{d}t$$
$$+ \int_0^T\!\!\int_\Omega \big(\rho u \otimes u - b \otimes b\big) : \nabla U\, \mathrm{d}x\, \mathrm{d}t + \int_0^T\!\!\int_\Omega \rho u \cdot U^\perp\, \mathrm{d}x\, \mathrm{d}t - \int_0^T\!\!\int_\Omega \nu(\rho) \nabla u : \nabla U\, \mathrm{d}x\, \mathrm{d}t\,.$$

Next, testing the mass equation in (5.12) against $|U|^2/2$ yields

(5.23)
$$\frac{1}{2}\int_\Omega \rho(T) |U(T)|^2\, \mathrm{d}x = \frac{1}{2}\int_\Omega \rho_0 |U(0)|^2\, \mathrm{d}x + \int_0^T\!\!\int_\Omega \rho\, U \cdot \partial_t U\, \mathrm{d}x\, \mathrm{d}t + \frac{1}{2}\int_0^T\!\!\int_\Omega \rho\, u \cdot \nabla |U|^2\, \mathrm{d}x\, \mathrm{d}t$$
$$= \frac{1}{2}\int_\Omega \rho_0 |U(0)|^2\, \mathrm{d}x + \int_0^T\!\!\int_\Omega \rho\, U \cdot \partial_t U\, \mathrm{d}x\, \mathrm{d}t + \int_0^T\!\!\int_\Omega \rho\, u \otimes U : \nabla U\, \mathrm{d}x\, \mathrm{d}t\,.$$

Recall that the energy inequality (5.15) reads

$$(5.24)\quad \frac{1}{2}\int_\Omega \Big\{\rho(T) |u(T)|^2 + |b(T)|^2\Big\}\, \mathrm{d}x \le \frac{1}{2}\int_\Omega \Big\{\rho_0 |u_0|^2 + |b_0|^2\Big\}\, \mathrm{d}x$$
$$- \int_0^T \int_\Omega \Big\{\nu(\rho) |\nabla u|^2 + \mu(\rho) |\nabla \times b|^2\Big\}\, \mathrm{d}x\, \mathrm{d}t\,.$$

Now, let us deal with the magnetic field. We start by using $\zeta = B$ as a test function in the magnetic field equation (5.14): we get

$$(5.25)\quad \int_\Omega b(T) \cdot B(T)\, \mathrm{d}x = \int_\Omega b_0 \cdot B(0)\, \mathrm{d}x + \int_0^T \int_\Omega b \cdot \partial_t B\, \mathrm{d}x\, \mathrm{d}t$$
$$+ \int_0^T \int_\Omega \big\{u \otimes b - b \otimes u\big\} : \nabla B\, \mathrm{d}x\, \mathrm{d}t - \int_0^T \int_\Omega \mu(\rho) \big(\nabla \times b\big) \big(\nabla \times B\big)\, \mathrm{d}t\, \mathrm{d}t\,.$$

By analogy with what we have done for the velocity field above, we now use $\phi = |B|^2/2$ as a test function in the (trivial) transport equation $\partial_t 1 + \operatorname{div}(1\, u) = 0$: we find

$$(5.26)\quad \frac{1}{2}\int_\Omega |B(T)|^2\, \mathrm{d}x = \frac{1}{2}\int_\Omega |B(0)|^2\, \mathrm{d}x + \int_0^T \int_\Omega B \cdot \partial_t B\, \mathrm{d}x + \frac{1}{2}\int_0^T \int_\Omega u \cdot \nabla |B|^2\, \mathrm{d}x\, \mathrm{d}t$$
$$= \frac{1}{2}\int_\Omega |B(0)|^2\, \mathrm{d}x + \int_0^T \int_\Omega B \cdot \partial_t B\, \mathrm{d}x + \int_0^T \int_\Omega u \otimes B : \nabla B\, \mathrm{d}x\, \mathrm{d}t\,.$$

Finally, we take care of the density oscillation functions r and R. Testing equation (5.18) against the smooth R gives

$$(5.27)\quad \int_\Omega r(T) R(T)\, \mathrm{d}x = \int_\Omega r_0 R(0)\, \mathrm{d}x + \int_0^T \int_\Omega r \partial_t R\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_\Omega r u \cdot \nabla R\, \mathrm{d}x\, \mathrm{d}t\,.$$

Using again (5.18) and the fact that $\operatorname{div} u = 0$, we deduce that the L^2 norm of r must be preserved in time (this is the same property as (5.16) above). In other words,

$$\int_\Omega |r(T)|^2 \,\mathrm{d}x = \int_\Omega |r_0|^2 \,\mathrm{d}x \,. \tag{5.28}$$

On the other hand, repeating the computations which led to (5.26), we obtain

$$\frac{1}{2}\int_\Omega |R(T)|^2 \,\mathrm{d}x = \frac{1}{2}\int |R(0)|^2 \,\mathrm{d}x + \int_0^T \int_\Omega R\partial_t R \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega R u \cdot \nabla R \,\mathrm{d}x\,\mathrm{d}t \,. \tag{5.29}$$

At this point, for notational convenience let us define

$$\delta r := r - R\,, \qquad \delta u := u - U\,, \qquad \delta b := b - B\,.$$

Putting relations (5.22), (5.23), (5.24), (5.25), (5.26), (5.27), (5.28) and (5.29) all together, we eventually find

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \Big\{\rho(T)\,|\delta u(T)|^2 \,+\, |\delta b(T)|^2 \,+\, |\delta r(T)|^2\Big\} \,\leq\, \frac{1}{2}\int_\Omega \Big\{\rho_0\,|\delta u_0|^2 \,+\, |\delta b_0|^2 \,+\, |\delta r_0|^2\Big\} \\
&\qquad - \int_0^T\!\!\int_\Omega \nu(\rho)\,\nabla u : \nabla\delta u \,-\, \int_0^T\!\!\int_\Omega \mu(\rho)\,\big(\nabla\times b\big)\,\big(\nabla\times\delta b\big) \,-\, \int_0^T\!\!\int_\Omega \rho\, u\cdot U^\perp \\
&\qquad - \int_0^T\!\!\int_\Omega \Big\{\rho\,\delta u\cdot\partial_t U \,+\, \delta b\cdot\partial_t B \,+\, \delta r\,\partial_t R\Big\} \,-\, \int_0^T\!\!\int_\Omega \Big\{\rho\, u\otimes\delta u : \nabla U \,+\, \delta r\, u\cdot\nabla R\Big\} \\
&\qquad + \int_0^T\int_\Omega \Big\{-\, u\otimes\delta b : \nabla B \,+\, b\otimes b : \nabla U \,+\, b\otimes u : \nabla B\Big\}\,.
\end{aligned} \tag{5.30}$$

Next, we remark that we can write[4]

$$\rho\, u\otimes\delta u : \nabla U \,=\, \rho\,\big(u\cdot\nabla\big)U\cdot\delta u \qquad \text{and} \qquad u\otimes\delta b : \nabla B \,=\, \big(u\cdot\nabla\big)B\cdot\delta b\,,$$

and that, by orthogonality, we have $u\cdot U^\perp \,=\, \delta u\cdot U^\perp$. Therefore, the right-hand side of (5.30) can be recasted as

$$\begin{aligned}
&\mathcal{E}\Big([r_0,u_0,b_0]\,\Big|\,[R(0),U(0),B(0)]\Big) \\
&- \int_0^T\!\!\int_\Omega \rho\,\Big(\partial_t U + (u\cdot\nabla)U + U^\perp\Big)\cdot\delta u - \int_0^T\!\!\int_\Omega \Big(\partial_t B + (u\cdot\nabla)B\Big)\cdot\delta b - \int_0^T\!\!\int_\Omega \Big(\partial_t R + u\cdot\nabla R\Big)\cdot\delta r \\
&- \int_0^T\!\!\int_\Omega \nu(\rho)\,\nabla u : \nabla\delta u - \int_0^T\!\!\int_\Omega \mu(\rho)\,\big(\nabla\times b\big)\,\big(\nabla\times\delta b\big) + \int_0^T\!\!\int_\Omega \Big\{b\otimes b : \nabla U \,+\, b\otimes u : \nabla B\Big\}\,.
\end{aligned}$$

Let us focus on the last two terms for a while. Simple manipulations show that

$$\begin{aligned}
b\otimes b : \nabla U \,&=\, \big(b\cdot\nabla\big)U\cdot b \,=\, \big(b\cdot\nabla\big)U\cdot\delta b \,+\, \big(b\cdot\nabla\big)U\cdot B \\
b\otimes u : \nabla B \,&=\, \big(b\cdot\nabla\big)B\cdot u \,=\, \big(b\cdot\nabla\big)B\cdot\delta u \,+\, \big(b\cdot\nabla\big)B\cdot U\,.
\end{aligned}$$

Observe that $\big(b\cdot\nabla\big)U\cdot B + \big(b\cdot\nabla\big)B\cdot U \,=\, b\cdot\nabla\big(B\cdot U\big)$. Therefore, owing to the the divergence-free condition on b, the previous relations imply that

$$\int_0^T\!\!\int_\Omega \Big\{b\otimes b : \nabla U \,+\, b\otimes u : \nabla B\Big\} \,=\, \int_0^T\!\!\int_\Omega \Big\{\big(b\cdot\nabla\big)U\cdot\delta b \,+\, \big(b\cdot\nabla\big)B\cdot\delta u\Big\}\,.$$

This completes the proof of (5.20) for smooth compactly supported (R,U,B).

[4] Notice that we have set $f\otimes g : \nabla h = \sum_{j,k} f_j g_k \partial_j h_k$; this corresponds to the agreement that $[\nabla h]_{ij} = \partial_i h_j$ is the transpose matrix of the differential of h.

In order to get the result for triplets (R, U, B) having the regularity stated in Theorem 5.10, we argue by density.

Firstly, we observe that, for the left-hand side of (5.20) to be well defined, it is enough to have

$$|U|^2 + |B|^2 + R^2 \in L^\infty_T(L^1) \qquad \text{and} \qquad \nabla U\,,\ \nabla B \in L^2_T(L^2)\,. \tag{5.31}$$

Next, we consider each term appearing in the definition (5.21) of $\mathcal{R}$. In view of a possible application of Grönwall lemma, in the first three terms it is natural to put $\sqrt{\rho}\,\delta u$, δB and δr in $L^\infty_T(L^2)$, so that, since $\rho \in L^\infty(\mathbb{R}_+ \times \Omega)$, one needs

$$\partial_t U + (u \cdot \nabla)U + U^\perp\,, \quad \partial_t B + (u \cdot \nabla)B\,, \quad \partial_t R + (u \cdot \nabla)R \qquad \text{to belong to} \qquad L^1_T(L^2)\,.$$

Owing to the regularity of $u \in L^2_T(H^1)$ and Sobolev embeddings, it is enough to require, besides the above conditions (5.31), also the conditions

$$\partial_t U\,,\ \partial_t B\,,\ \partial_t R \in L^1_T(L^2) \qquad \text{and} \qquad \nabla U\,,\ \nabla B\,,\ \nabla R \in L^2_T(L^q)\,,$$

for some $q > 2$. As for the last two terms appearing in (5.21), conditions (5.31) are enough. Finally, for the last integral appearing in the second line of (5.21), we remark that, by Gagliardo-Nirenberg inequality of Lemma 5.12, we deduce the property $b \in L^{2p/(p-2)}_T(L^p)$ for any $2 \leq p < +\infty$, and the same actually holds true also for δu and δb. Taking $p = 4$, we infer that b, δu and δb belong to $L^4_T(L^4)$, so that conditions (5.31) are also enough for treating those terms.

The proof to Theorem 5.10 is now completed.

5.4 Derivation of Quasi-Homogeneous MHD Systems in 2-D

This section is devoted to the proof of Theorems 5.4 and 5.7, concerning the rigorous derivation of systems (5.7) and (5.10) respectively, and carried out respectively in Subsections 5.4.1 and 5.4.2 below. The main tool will be the relative entropy inequality for the primitive system, proved in the previous section.

Notice that the proof in the viscous and resistive case is actually more delicate than in the ideal case, inasmuch as one disposes of less regularity on the limit points (R, U, B), which act as the smooth functions to be used in the relative entropy inequality (5.19).

5.4.1 Derivation of the Viscous and Resistive System

In this subsection we carry out the proof of Theorem 5.4. Our argument is in three steps. First, we equip ourselves with a solution (R, UB) of the target system by resorting to a previously proved well-posedness result [27]. Next, we write the relative entropy estimate for solutions $(r_\epsilon, u_\epsilon, b_\epsilon)$ of the primitive equations and (R, U, B). And finally, we will have to estimate all the integrals involved in the remainder term so as to obtain a differential inequality to which Grönwall's lemma may be applied.

To begin with, we need the following result (see Theorem 5.3 in [27] for the proof), which makes quantitative the regularity properties of the solutions to system (5.7), stated in Theorem 5.3.

Proposition 5.11. *Let $\beta \in]0,1[$ and $(R_0, U_0, B_0) \in H^{1+\beta} \times H^1 \times H^1$ be a set of initial data, and consider (R, U, B) the corresponding unique solution to system* (5.7) *provided by Theorem 5.3. Then the following estimates hold true:*

(i) the basic energy inequality*: for any $t \geq 0$, one has*

$$\frac{1}{2}\big\|\big(U(t), B(t)\big)\big\|_{L^2} + \int_0^t \int_\Omega \Big(\nu(1)\|\nabla U\|^2_{L^2} + \mu(1)\|\nabla B\|_{L^2}\Big)\,\mathrm{d}x\,\mathrm{d}\tau \leq \frac{1}{2}\big\|(U_0, B_0)\big\|_{L^2}\,;$$

(ii) *the basic* transport estimates*: for any $p \in [2, +\infty]$ and any $t \geq 0$, one has*

$$\|R(t)\|_{L^p} = \|R_0\|_{L^p}\,;$$

(iii) *for any exponent $h > 0$, there is a constant $C = C\big(\|R_0\|_{L^2\cap L^\infty}, \|(U_0, B_0)\|_{H^1}, h\big) > 0$ such that, for any time $T > 0$, one has*

$$\big\|\nabla U\big\|_{L^\infty_T(L^2)} + \big\|(\Delta U, \partial_t U)\big\|_{L^2_T(L^2)} + \big\|\nabla B\big\|_{L^\infty_T(L^2)} + \big\|(\Delta B, \partial_t B)\big\|_{L^2_T(L^2)} \leq C\left(1 + T^h\right)\,;$$

(iv) *for any $0 < \gamma < \beta$, there exists $C = C(\gamma, \beta) > 0$ such that, any fixed time $T > 0$, one has*

$$\|R\|_{L^\infty_T(H^\gamma)} \leq C\|R_0\|_{H^\beta} \exp\left\{C\left(\int_0^T \|\nabla U\|_{H^{1+\gamma}}\,\mathrm{d}t\right)^2\right\}.$$

Now we can proceed to the proof of Theorem 5.4. As already mentioned, the main tool here is the relative entropy inequality established in Theorem 5.10 above.

Using the Relative Entropy Inequality

Owing to the properties stated in Theorem 5.3 and standard product rules in Sobolev spaces (see Propositions 1.20 and 1.21), one can see that the solution $\big(R, U, B\big)$ to the limit problem (5.7) meets the regularity requirements of Theorem 5.10. Thus, it can be employed as a test function in the relative entropy functional (5.19).

Therefore, the relative entropy inequality (5.20) yields, for all $T > 0$, the estimate

$$\mathcal{E}\Big([r_\varepsilon, u_\varepsilon, b_\varepsilon]\,\Big|\,[R, U, B]\Big)(T) + \int_0^T\!\!\int_\Omega \Big(\nu(\rho_\varepsilon)\left|\nabla\delta u_\varepsilon\right|^2 + \mu(\rho_\varepsilon)\left|\nabla\times\delta b_\varepsilon\right|^2\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \mathcal{E}\Big([r_{0,\varepsilon}, u_{0,\varepsilon}, b_{0,\varepsilon}]\,\Big|\,[R_0, U_0, B_0]\Big) + \int_0^T \mathcal{R}_\varepsilon\,\mathrm{d}t\,,$$

where the reminder term $\mathcal{R}_\varepsilon := \mathcal{R}\big(r_\varepsilon, u_\varepsilon, b_\varepsilon; R, U, B\big)$ is defined by the formula

$$\begin{aligned}\mathcal{R}_\varepsilon = \sum_{j=1}^7 I_j = &-\int_\Omega \rho_\varepsilon\left(\partial_t U + (u_\varepsilon\cdot\nabla)U + \frac{1}{\varepsilon}U^\perp\right)\cdot\delta u_\varepsilon\,\mathrm{d}x - \int_\Omega\big(\partial_t B + (u_\varepsilon\cdot\nabla)B\big)\cdot\delta b_\varepsilon\,\mathrm{d}x\\ &-\int_\Omega\big(\partial_t R + u_\varepsilon\cdot\nabla R\big)\cdot\delta r_\varepsilon\,\mathrm{d}x + \int_\Omega\big(b_\varepsilon\cdot\nabla\big)U\cdot\delta b_\varepsilon\,\mathrm{d}x + \int_\Omega\big(b_\varepsilon\cdot\nabla\big)B\cdot\delta u_\varepsilon\,\mathrm{d}x\\ &-\int_\Omega \nu(\rho_\varepsilon)\,\nabla U : \nabla\delta u_\varepsilon\,\mathrm{d}x - \int_\Omega \mu(\rho_\varepsilon)\,\big(\nabla\times B\big)\,\big(\nabla\times\delta b_\varepsilon\big)\,\mathrm{d}x\,.\end{aligned}$$

At this point, we use the fact that $\big(R, U, B\big)$ is a strong solution of the limit system (5.7). We start by focusing on the density term R: using the first relation in (5.7) yields

$$\partial_t R + u_\varepsilon\cdot\nabla R = -U\cdot\nabla R + u_\varepsilon\cdot\nabla R = \delta u_\varepsilon\cdot\nabla R\,. \tag{5.32}$$

Similarly, for the magnetic field we get

$$\partial_t B + (u_\varepsilon\cdot\nabla)B = (\delta u_\varepsilon\cdot\nabla)B + (B\cdot\nabla)U + \mu(1)\Delta B\,.$$

Observe that $\Delta B = \nabla^\perp\nabla\times B$, owing to the divergence-free condition on B. Therefore, putting together I_2, I_4 and I_7, thanks to the previous equality we can write

$$I_2 + I_4 + I_7 = \int_\Omega \big(\delta b_\varepsilon\cdot\nabla\big)U\cdot\delta b_\varepsilon - \int_\Omega (\delta u_\varepsilon\cdot\nabla)B\cdot\delta b_\varepsilon \tag{5.33}$$

$$- \int_\Omega \big(\mu(\rho_\varepsilon) - \mu(1)\big)\big(\nabla \times B\big)\,\big(\nabla \times \delta b_\varepsilon\big)\,,$$

where we have also used an integration by parts for the term presenting ΔB.

It remains us to deal with the velocity field. First of all, because of the decomposition $\rho_\varepsilon = 1 + \varepsilon r_\varepsilon$, the term I_1 can be recasted as

$$\begin{aligned}
I_1 \;&=\; -\int_\Omega \left(\partial_t U + (u_\varepsilon \cdot \nabla)U + \frac{1}{\varepsilon}U^\perp\right)\cdot \delta u_\varepsilon - \varepsilon\int_\Omega r_\varepsilon \left(\partial_t U + (u_\varepsilon \cdot \nabla)U + \frac{1}{\varepsilon}U^\perp\right)\cdot \delta u_\varepsilon \\
&=\; -\int_\Omega \big(\partial_t U + (u_\varepsilon \cdot \nabla)U\big)\cdot \delta u_\varepsilon \;-\; \varepsilon\int_\Omega r_\varepsilon \left(\partial_t U + (u_\varepsilon \cdot \nabla)U + \frac{1}{\varepsilon}U^\perp\right)\cdot \delta u_\varepsilon \\
&=\; I_{11} + I_{12}\,.
\end{aligned} \tag{5.34}$$

Notice that, in passing from the first to the second line, we have got rid of the singular term $\varepsilon^{-1}U^\perp$ in the first integral. Indeed, the condition $\operatorname{div} U = 0$ implies that $U = \nabla^\perp\theta$, for some suitable stream function θ. Hence $U^\perp$ is a perfect gradient, and this, together with the condition $\operatorname{div}\delta u_\varepsilon = 0$, implies that

$$\int_\Omega U^\perp \cdot \delta u_\varepsilon \;=\; -\int_\Omega \nabla\theta \cdot \delta u_\varepsilon \;=\; 0\,.$$

In view of this property, let us come back to (5.34). We us focus on I_{11} for a while. For dealing with this term, we can use the second equation in (5.7) to write

$$I_{11} \;=\; -\int_\Omega \Big((\delta u_\varepsilon \cdot)\nabla U - RU^\perp + \nu(1)\Delta U + (B\cdot\nabla)B\Big)\cdot \delta u_\varepsilon\,,$$

where, once again, we have omitted to write the terms which are pure gradients, since $\operatorname{div}\delta u_\varepsilon = 0$. Combining this term with I_5 and I_6, we find, after integration by parts

$$I_{11} \,+\, I_5 \,+\, I_6 \;=\; -\int_\Omega \Big((\delta u_\varepsilon \cdot \nabla)U - (\delta b_\varepsilon\cdot\nabla)B - RU^\perp\Big)\cdot \delta u_\varepsilon \;-\; \int_\Omega \big(\nu(\rho_\varepsilon) - \nu(1)\big)\nabla U : \nabla\delta u_\varepsilon\,.$$

Plugging the expression of I_{12} into this equation, we finally get

$$\begin{aligned}
I_1 \,+\, I_5 \,+\, I_6 \;=\; &-\int_\Omega \Big((\delta u_\varepsilon \cdot \nabla)U - (\delta b_\varepsilon\cdot\nabla)B - \delta r_\varepsilon U^\perp\Big)\cdot \delta u_\varepsilon \\
&- \int_\Omega \big(\nu(\rho_\varepsilon) - \nu(1)\big)\nabla U : \nabla\delta u_\varepsilon - \varepsilon\int_\Omega r_\varepsilon\big(\partial_t U + (u_\varepsilon\cdot\nabla)U\big)\cdot\delta u_\varepsilon\,.
\end{aligned} \tag{5.35}$$

In the end, by use of (5.32), (5.33) and (5.35), we deduce that

$$\begin{aligned}
\mathcal{R}_\varepsilon \;=\; &-\int_\Omega \delta r_\varepsilon\,\delta u_\varepsilon\cdot\nabla R + \int_\Omega \Big((\delta b_\varepsilon\cdot\nabla)U - (\delta u_\varepsilon\cdot\nabla)B\Big)\cdot\delta b_\varepsilon \\
&- \int_\Omega \big(\mu(\rho_\varepsilon) - \mu(1)\big)\big(\nabla\times B\big)\,\big(\nabla\times\delta b_\varepsilon\big) \\
&- \int_\Omega \Big((\delta u_\varepsilon \cdot \nabla)U - (\delta b_\varepsilon\cdot\nabla)B - \delta r_\varepsilon U^\perp\Big)\cdot \delta u_\varepsilon \\
&- \int_\Omega \big(\nu(\rho_\varepsilon) - \nu(1)\big)\nabla U : \nabla\delta u_\varepsilon \,-\, \varepsilon\int_\Omega r_\varepsilon\big(\partial_t U + (u_\varepsilon\cdot\nabla)U\big)\cdot\delta u_\varepsilon \;=\; \sum_{\ell=1}^{6} J_\ell\,.
\end{aligned} \tag{5.36}$$

Our next goal is to bound all the terms J_ℓ appearing in the previous expression: this is the goal of the next paragraph. Notice that those bounds will complete in fact the proof to Theorem 5.4. Indeed, since the density is a perturbation of a constant state, i.e. $\rho_\epsilon = 1+\epsilon r_\epsilon$, the relative entropy $\mathcal{E}\Big([r_\varepsilon, u_\varepsilon, b_\varepsilon]\,\Big|\,[R,U,B]\Big)$ is in fact equivalent to the L^2 norm of the error function $(\delta r_\varepsilon, \delta u_\varepsilon, \delta b_\varepsilon)$.

Quantitative Estimates for the Viscous Resistive System

In the computations below, we make extensive use of the Gagliardo-Nirenberg inequality (GN for short), which we reproduce here below. We refer to Corollary 1.2 of [21] for a proof.

Lemma 5.12. *Let $f \in H^1$ and $p \in [2, +\infty[$ such that $1/p > 1/2 - 1/d$. Then*

$$\|f\|_{L^p} \leq C(p) \|f\|_{L^2}^{1-\lambda} \|\nabla f\|_{L^2}^{\lambda}, \qquad \textit{with } \lambda = \frac{d(p-2)}{2p}.$$

In particular, if $d = 2$, then $\|f\|_{L^p} \leq C(p) \|f\|_{L^2}^{2/p} \|\nabla f\|_{L^2}^{1-2/p}$.

According to the notation introduced above, we agree to note $M_p(t) \in L^p(\mathbb{R}_+)$ generic globally L^p functions; on the other hand, we will use the notation $N_p(t)$ to denote functions in $L^p_{\mathrm{loc}}(\mathbb{R}_+)$.

Let us bound all the terms appearing in (5.36). We start by handling J_1. Recall that $R \in L^\infty_T(H^{1+\gamma})$ for any $0 \leq \gamma < \beta$. Making use of Hölder's and GN inequalities, we get

$$\begin{aligned}\left| \int_\Omega (\nabla R \cdot \delta u_\varepsilon) \delta r_\varepsilon \, \mathrm{d}x \right| &\leq \|\delta r_\varepsilon\|_{L^2} \|\delta u_\varepsilon\|_{L^p} \|\nabla R\|_{L^q} \leq C \|\nabla R\|_{L^q} \|\delta r_\varepsilon\|_{L^2} \|\delta u_\varepsilon\|_{L^2}^{2/p} \|\nabla(\delta u_\varepsilon)\|_{L^2}^{1-2/p} \\ &\leq \eta \|\nabla(\delta u_\varepsilon)\|_{L^2}^2 + C(\eta, q) \|\nabla R\|_{L^q}^{q'} \|\delta r_\varepsilon\|_{L^2}^{q'} \|\delta u_\varepsilon\|_{L^2}^{2q'/p},\end{aligned}$$

where $\eta > 0$ is arbitrarily small, $p, q \geq 2$ are chosen so that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and the exponent q' is associated to q in Young's inequality by $\frac{1}{q} + \frac{1}{q'} = 1$. Using Young's inequality one more time with the exponents $\alpha = \frac{2(q-1)}{q}$ and $\beta = \frac{2(q-1)}{q-2}$ (which satisfy $\frac{1}{\alpha} + \frac{1}{\beta} = 1$) allows us to introduce the relative entropy function $\mathcal{E}(t)$ in the right-hand side:

$$\left| \int_\Omega (\nabla R \cdot \delta u_\varepsilon) \delta r_\varepsilon \, \mathrm{d}x \right| \leq \eta \|\nabla(\delta u_\varepsilon)\|_{L^2}^2 + C(\eta, q) \left(1 + \|\nabla R\|_{L^q}^2\right) \mathcal{E}(t).$$

Now since $\nabla R \in L^\infty_T(H^\gamma)$, we see that $\nabla R \in L^\infty_T(L^q)$ for q close enough to 2 by Sobolev embedding. For such q, it is always possible to find a $p \geq 2$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, so that all of the preceding inequalities are justified. *In fine*, using Proposition 5.11, we find the following inequality:

$$\left| \int_\Omega (\nabla R \cdot \delta u_\varepsilon) \delta r_\varepsilon \, \mathrm{d}x \right| \leq \eta \|\nabla(\delta u_\varepsilon)\|_{L^2}^2 + C(\eta, q, \|r_0\|_{H^{1+\beta}}, T) \mathcal{E}(t). \tag{5.37}$$

Next, we look at J_2. Using Hölder's inequality with exponents $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$, we get

$$\left| \int_\Omega (\delta b_\varepsilon \cdot \nabla) U \cdot \delta b_\varepsilon \, \mathrm{d}x \right| \leq \|\nabla U\|_{L^2} \|\delta b_\varepsilon\|_{L^4}^2 \leq \|\nabla U\|_{L^2} \|\delta b_\varepsilon\|_{L^2} \|\nabla \delta b_\varepsilon\|_{L^2},$$

where we have also exploited GN inequality. Using now Young's inequality, we infer, for all $\eta > 0$, the bound

$$\begin{aligned}\left| \int_\Omega (\delta b_\varepsilon \cdot \nabla) U \cdot \delta b_\varepsilon \, \mathrm{d}x \right| &\leq \eta \|\nabla \delta b_\varepsilon\|_{L^2}^2 + C(\eta) \|\nabla U\|_{L^2}^2 \|\delta b_\varepsilon\|_{L^2}^2 \\ &= \eta \|\nabla \delta b_\varepsilon\|_{L^2}^2 + M_1(t) \|\delta b_\varepsilon\|_{L^2}^2,\end{aligned}$$

where we have set $M_1(t) = C(\eta) \|\nabla U(t)\|_{L^2}^2$. Notice that, thanks to the estimates of Proposition 5.11, one has $M_1 \in L^1(\mathbb{R}_+)$, with $\|M_1\|_{L^1(\mathbb{R}_+)} = C(\eta, \|u_0\|_{L^2}, \|b_0\|_{L^2})$. In the very same way, we also have

$$\begin{aligned}\left| \int_\Omega (\delta u_\varepsilon \cdot \nabla) B \cdot \delta b_\varepsilon \, \mathrm{d}x \right| &\leq \|\nabla B\|_{L^2} \|\delta u_\varepsilon\|_{L^4} \|\delta b_\varepsilon\|_{L^4} \\ &\leq C \|\nabla B\|_{L^2} \|\delta u_\varepsilon\|_{L^2}^{1/2} \|\nabla \delta u_\varepsilon\|_{L^2}^{1/2} \|\delta b_\varepsilon\|_{L^2}^{1/2} \|\nabla \delta b_\varepsilon\|_{L^2}^{1/2} \\ &\leq \eta \Big(\|\nabla \delta u_\varepsilon\|_{L^2}^2 + \|\nabla \delta b_\varepsilon\|_{L^2}^2 \Big) + C(\eta) \|\nabla B\|_{L^2}^2 \Big(\|\delta u_\varepsilon\|_{L^2}^2 + \|\delta b_\varepsilon\|_{L^2}^2 \Big) \\ &= \eta \Big(\|\nabla \delta u_\varepsilon\|_{L^2}^2 + \|\nabla \delta b_\varepsilon\|_{L^2}^2 \Big) + M_1(t) \mathcal{E}(t),\end{aligned}$$

where $M_1(t) = \|\nabla B(t)\|_{L^2}$ is, as above, an $L^1(\mathbb{R}_+)$ function whose L^1 norm is bounded by a constant $C = C(\eta, \|u_0\|_{L^2}, \|b_0\|_{L^2})$. In the end, we have proved the following bound for J_2:

$$|J_2| \leq 2\eta\Big(\|\nabla\delta u_\varepsilon\|^2_{L^2} + \|\nabla\delta b_\varepsilon\|^2_{L^2}\Big) + M_1(t)\mathcal{E}(t)\,, \tag{5.38}$$

where $\eta > 0$ can be chosen arbitrarily small.

We now consider J_3. By assumption, μ is σ-continuous in a neighborhood of 1, with σ being non-decreasing. Therefore, for $\epsilon > 0$ so small that $\epsilon M \leq 1$, with M defined in the statement of Theorem 5.4, we can estimate

$$\begin{aligned}\left|\int_\Omega \big(\mu(\rho_\epsilon) - \mu(1)\big)(\nabla\times B)\cdot(\nabla\times\delta b_\varepsilon)\right| &\leq |\mu|_{C_\sigma}\sigma\left(\epsilon\|r_\epsilon\|_{L^\infty}\right)\|\nabla\times B\|_{L^2}\|\nabla\times\delta b_\varepsilon\|_{L^2} \\ &\leq \eta\|\nabla\times\delta b_\varepsilon\|^2_{L^2} + C(\eta, |\mu|_{C_\sigma})\sigma(M\epsilon)^2\|\nabla\times B\|^2_{L^2} \\ &= \eta\|\nabla\times\delta b_\varepsilon\|^2_{L^2} + \sigma(M\epsilon)^2 M_1(t)\,,\end{aligned} \tag{5.39}$$

where $M_1 \in L^1(\mathbb{R}_+)$, with (in view of Proposition 5.11) $\|M_1\|_{L^1(\mathbb{R}_+)}$ depending only on η, $|\mu|_{C_\sigma}$, $\|u_0\|_{L^2}$ and $\|b_0\|_{L^2}$. The integral J_5 containing the viscosity term is dealt with in the same way.

For J_4, we separate the integral into three summands:

$$J_4 = \int_\Omega (\delta b_\varepsilon\cdot\nabla)B\cdot\delta u_\varepsilon\,\mathrm{d}x - \int_\Omega(\delta u_\varepsilon\cdot\nabla)U\cdot\delta u_\varepsilon\,\mathrm{d}x - \int_\Omega \delta r_\varepsilon U^\perp\cdot\delta u_\varepsilon\,\mathrm{d}x := J_{4,1} + J_{4,2} + J_{4,3}\,.$$

The first term $J_{4,1}$ can be dealt with in a way analogous to the second term appearing in J_2: combining Hölder and GN inequalities, we get

$$\left|\int_\Omega(\delta b_\varepsilon\cdot\nabla)B\cdot\delta u_\varepsilon\,\mathrm{d}x\right| \leq \|\delta u_\varepsilon\|_{L^4}\|\delta b_\varepsilon\|_{L^4}\|\nabla B\|_{L^2} \leq \eta\Big(\|\nabla\delta u_\varepsilon\|^2_{L^2} + \|\nabla\delta b_\varepsilon\|^2_{L^2}\Big) + M_1(t)\mathcal{E}(t)\,,$$

where $M_1 \in L^1(\mathbb{R}_+)$, with $\|M_1\|_{L^1(\mathbb{R}_+)}$ being a function of $(\eta, \|u_0\|_{L^2}, \|b_0\|_{L^2})$. Now look at $J_{4,2}$: a very similar argument yields, for $M_1(t) = C(\eta)\|\nabla U(t)\|^2_{L^2}$, with $M_1 \in L^1(\mathbb{R}_+)$, the inequality

$$\left|\int_\Omega(\delta u_\varepsilon\cdot\nabla)U\cdot\delta u_\varepsilon\,\mathrm{d}x\right| \leq \eta\|\nabla(\delta u_\varepsilon)\|^2_{L^2} + M_1(t)\mathcal{E}(t)\,.$$

Finally, we notice that $J_{4,3}$ is very similar to J_1, up to substituting ∇R with U. In fact, since $U \in L^\infty_T(H^1)$ (see Proposition 5.11), and not only $L^\infty_T(H^\gamma)$ as ∇R before, it suffices to conduct the computations for any values of p and q: taking for simplicity $p = q = 4$, we deduce

$$\begin{aligned}\left|\int_\Omega \delta r_\varepsilon U^\perp\cdot\delta u_\varepsilon\,\mathrm{d}x\right| &\leq \|U\|_{L^4}\|\delta r_\varepsilon\|_{L^2}\|\delta u_\varepsilon\|_{L^4} \leq \eta\|\nabla(\delta u_\varepsilon)\|^2_{L^2} + C(\eta)\|U\|^{4/3}_{L^\infty_T(H^1)}\|\delta r_\varepsilon\|^{4/3}_{L^2}\|\delta u_\varepsilon\|^{2/3}_{L^2} \\ &\leq \eta\|\nabla(\delta u_\varepsilon)\|^2_{L^2} + C(\eta, \|u_0\|_{H^1}, \|b_0\|_{H^1}, T)\,\mathcal{E}(t)\,.\end{aligned}$$

Summing up all the last inequalities, we gather

$$|J_4| \leq 3\eta\Big(\|\nabla\delta u_\varepsilon\|^2_{L^2} + \|\nabla\delta b_\varepsilon\|^2_{L^2}\Big) + \big(M_1(t) + C\big)\mathcal{E}(t)\,, \tag{5.40}$$

where, for simplicity, we have omitted the various quantities on which the constant $C > 0$ depends.

It remains to bound J_6. On the one hand, the integral containing the time derivative can be bounded by using Proposition 5.11:

$$\begin{aligned}\epsilon\left|\int_\Omega r_\epsilon\partial_t U\cdot\delta u_\varepsilon\,\mathrm{d}x\right| &\leq \epsilon\|\partial_t U\|_{L^2}\|r_\epsilon\|_{L^\infty}\|\delta u_\varepsilon\|_{L^2} \leq \epsilon^2 C(\|r_{0,\epsilon}\|_{L^\infty})\|\partial_t U\|^2_{L^2} + \mathcal{E}(t) \\ &\leq \epsilon^2 N_1(t) + \mathcal{E}(t),\end{aligned}$$

where $\|N_1\|_{L^1_T}$ grows at polynomial speed $1+T^h$ and depends on $(h, \|r_{0,\epsilon}\|_{L^\infty}, \|u_0\|_{H^1}, \|b_0\|_{H^1}, T)$. On the other hand, using Hölder's and GN inequalities we infer

$$\begin{aligned}\left|\epsilon\int_\Omega r_\epsilon(u_\epsilon\cdot\nabla)U\cdot\delta u_\varepsilon\,\mathrm{d}x\right| &\leq \epsilon\|r_\epsilon\|_{L^\infty}\|u_\epsilon\|_{L^4}\|\nabla U\|_{L^2}\|\delta u_\varepsilon\|_{L^4}\\ &\leq \epsilon C(\|r_{0,\epsilon}\|_{L^\infty})\|u_\epsilon\|_{L^2}^{1/2}\|\nabla u_\epsilon\|_{L^2}^{1/2}\|\nabla U\|_{L^2}\|\delta u_\varepsilon\|_{L^2}^{1/2}\|\nabla(\delta u_\varepsilon)\|_{L^2}^{1/2}\end{aligned}$$

Recall that $u_\epsilon\in L^\infty(L^2)$. Using Young's inequality a first time with coefficients $\frac14+\frac34=1$,

$$\left|\epsilon\int_\Omega r_\epsilon(u_\epsilon\cdot\nabla)U\cdot\delta u_\varepsilon\,\mathrm{d}x\right| \leq \eta\|\nabla(\delta u_\varepsilon)\|_{L^2}^2+\epsilon^{4/3}C\|\nabla u_\epsilon\|_{L^2}^{2/3}\|\nabla U\|_{L^2}^{4/3}\|\delta u_\varepsilon\|_{L^2}^{2/3},$$

with $C=C(\eta,\|r_{0,\epsilon}\|_{L^\infty},\|u_{0,\epsilon}\|_{L^2},\|b_{0,\epsilon}\|_{L^2})$, and a second time on the second summand with coefficients $\frac13+\frac23=1$, we gather

$$\begin{aligned}\left|\epsilon\int_\Omega r_\epsilon(u_\epsilon\cdot\nabla)U\cdot\delta u_\varepsilon\,\mathrm{d}x\right| &\leq \eta\|\nabla(\delta u_\varepsilon)\|_{L^2}^2+\|\delta u_\varepsilon\|_{L^2}^2\|\nabla u_\epsilon\|_{L^2}^2+\epsilon^2C\|\nabla U\|_{L^2}^2\\ &\leq \eta\|\nabla(\delta u_\varepsilon)\|_{L^2}^2+M_1(t)\mathcal{E}(t)+\epsilon^2M_1(t),\end{aligned}$$

where $C>0$ depends on the same quantities as the previous constant, and we have used the fact that both ∇u_ϵ and ∇U belong to $L^2(\mathbb{R}_+;L^2)$ to introduce the function $M_1\in L^1(\mathbb{R}_+)$. Notice that the L^1 norm of M_1 depends on $(\eta,\|r_{0,\epsilon}\|_{L^\infty},\|u_{0,\epsilon}\|_{L^2},\|b_{0,\epsilon}\|_{L^2})$. In the end, we deduce that

$$|J_6|\,\leq\,\eta\Big(\|\nabla\delta u_\varepsilon\|_{L^2}^2+\|\nabla\delta b_\varepsilon\|_{L^2}^2\Big)+\big(M_1(t)\,+\,1\big)\mathcal{E}(t)\,+\,\epsilon^2\,\big(M_1(t)\,+\,N_1(t)\big)\,. \tag{5.41}$$

Piecing inequalities (5.37), (5.38), (5.39), (5.40), (5.41) all together and taking η small enough, say $\eta=\frac{1}{100}\min\{\nu_*,\mu_*\}$, we find

$$\begin{aligned}&\mathcal{E}(T)+\int_0^T\int_\Omega\Big\{\nu_*|\nabla\delta u_\varepsilon|^2+\mu_*|\nabla\delta b_\varepsilon|^2\Big\}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\leq C\left(\mathcal{E}\Big([r_{0,\varepsilon},u_{0,\varepsilon},b_{0,\varepsilon}]\,\Big|\,[R_0,U_0,B_0]\Big)+\int_0^T\Big(\mathcal{M}_1(t)\mathcal{E}(t)+\max\big\{\epsilon^2\,,\,\sigma^2(M\epsilon)\big\}\mathcal{N}_1(t)\Big)\mathrm{d}t\right)\end{aligned}$$

for any $T>0$, with $\mathcal{M}_1$ and $\mathcal{N}_1$ being locally integrable functions on $\mathbb{R}_+$. Use of Grönwall's lemma on this differential inequality provides the result we covet, namely inequality (5.8).

The proof of Theorem 5.4 is thus completed.

5.4.2 Vanishing Viscosity and Resistivity Limit: Derivation of the Ideal System

In this subsection, we show the proof of Theorem 5.7, concerning the derivation of the ideal system, which corresponds to the case $h(\epsilon)\to0^+$. With respect to the previous case, we lose any control on the gradients of the quantities δu_ϵ and δb_ϵ, since we have to deal with a vanishing viscosity and resistivity limit. On the other hand, the solution (R,U,B) to the limit problem will enjoy, on its lifespan, much more smoothness than in the previous section. In addition, we point out that the convergence here is limited to the time T^* representing the lifespan of (R,U,B), which is possibly finite.

Also in this section, the main ingredient is the relative entropy inequality of Theorem 5.10. We skip the proof of the fact that (R,U,B) verifies indeed the regularity hypotheses of that statement.

So, let us write the relative entropy inequality (5.20) for (r,u,b) and (R,U,B): we get

$$
\begin{aligned}
(5.42)\quad \mathcal{E}\Big([r_\epsilon, u_\epsilon, b_\epsilon]\big|[R,U,B]\Big)(T) + h(\epsilon)\int_0^T\int_\Omega \Big\{\nu(\rho_\epsilon)|\nabla\delta u_\varepsilon|^2 + \mu(\rho_\epsilon)|\nabla\times\delta b_\varepsilon|^2\Big\}\,\mathrm{d}x\,\mathrm{d}t \\
\leq \mathcal{E}\Big([r_{0,\varepsilon}, u_{0,\varepsilon}, b_{0,\varepsilon}]\,\Big|\,[R_0,U_0,B_0]\Big)\int_0^T \mathcal{R}\big(r_\epsilon, u_\epsilon, b_\epsilon; R,U,B\big)\,\mathrm{d}t.
\end{aligned}
$$

Performing exactly the same computations as in Paragraph 5.4.1, we get an expression for the reminder term analogous to (5.36):

$$
\begin{aligned}
\mathcal{R}_\varepsilon \,=\, & -\int_\Omega \delta r_\varepsilon\,\delta u_\varepsilon\cdot\nabla R + \int_\Omega \Big((\delta b_\varepsilon\cdot\nabla)U - (\delta u_\varepsilon\cdot\nabla)B\Big)\cdot\delta b_\varepsilon \\
& - h(\epsilon)\int_\Omega \mu(\rho_\varepsilon)\big(\nabla\times B\big)\,\big(\nabla\times\delta b_\varepsilon\big) \\
& - \int_\Omega \Big((\delta u_\varepsilon\cdot\nabla)U - (\delta b_\varepsilon\cdot\nabla)B - \delta r_\varepsilon U^\perp\Big)\cdot\delta u_\varepsilon \\
& - h(\epsilon)\int_\Omega \nu(\rho_\varepsilon)\nabla U : \nabla\delta u_\varepsilon \,-\, \varepsilon\int_\Omega r_\varepsilon\big(\partial_t U + (u_\varepsilon\cdot\nabla)U\big)\cdot\delta u_\varepsilon \,=\, \sum_{\ell=1}^{6} J_\ell\,.
\end{aligned}
$$

We are going to bound all the integrals $J_1,\ldots,J_6$ one after the other.

First of all, for J_1 we have

$$
\begin{aligned}
\left|\int_\Omega \delta r_\varepsilon\delta u_\varepsilon\cdot\nabla R\,\mathrm{d}x\right| &\leq \|\nabla R\|_{L^\infty}\|\delta r_\varepsilon\|_{L^2}\|\delta u_\varepsilon\|_{L^2} \leq \|R\|_{L^\infty_T(H^s)}\Big(\|\delta r_\varepsilon\|_{L^2}^2 + \|\delta u_\varepsilon\|_{L^2}^2\Big) \\
&\leq C\big(T, \|(R_0,U_0,B_0)\|_{H^s}\big)\mathcal{E}(t)\,.
\end{aligned}
$$

As for J_2, we argue in the very same way and use Sobolev inequality to get

$$
\begin{aligned}
\left|\int_\Omega \Big((\delta b_\varepsilon\cdot\nabla)U\cdot\delta b_\varepsilon - (\delta u_\varepsilon\cdot\nabla)B\cdot\delta b_\varepsilon\Big)\,\mathrm{d}x\right| &\leq \Big(\|\nabla U\|_{L^\infty} + \|\nabla B\|_{L^\infty}\Big)\Big(\|\delta u_\varepsilon\|_{L^2}^2 + \|\delta b_\varepsilon\|_{L^2}^2\Big) \\
&\leq C\big(T, \|(R_0,U_0,B_0)\|_{H^s}\big)\mathcal{E}(t)\,.
\end{aligned}
$$

The fourth integral J_4 is dealt with in the same manner:

$$
\begin{aligned}
&\left|\int_\Omega \Big((\delta u_\varepsilon\cdot\nabla)U + (\delta b_\varepsilon\cdot\nabla)B - \delta r_\varepsilon U^\perp\Big)\cdot\delta u_\varepsilon\,\mathrm{d}x\right| \\
&\qquad\leq \|\nabla U\|_{L^\infty}\|\delta u_\varepsilon\|_{L^2}^2 + \|\nabla B\|_{L^\infty}\|\delta u_\varepsilon\|_{L^2}\|\delta b_\varepsilon\|_{L^2} + \|U\|_{L^\infty}\|\delta r_\varepsilon\|_{L^2}\|\delta u_\varepsilon\|_{L^2} \\
&\qquad\leq C\big(T, \|(R_0,U_0,B_0)\|_{H^s}\big)\,\mathcal{E}(t)\,.
\end{aligned}
$$

Now we take care of the integrals containing the derivatives of the error functions δu_ε and δb_ε, namely J_3 and J_5. Here, we use the fact that $\nabla\delta u_\varepsilon$ and $\nabla\delta b_\varepsilon$ have L^2 regularity, even though it is not uniformly with respect to ϵ. More precisely, the energy inequality for the primitive system (5.1), see item (vii) of Definition 5.1, yields, for any $T>0$, the uniform bound

$$
\big\|\sqrt{h(\epsilon)}\;\nabla u_\epsilon\big\|_{L^2_T(L^2)} + \big\|\sqrt{h(\epsilon)}\;\mu(\rho_\epsilon)\nabla\times b_\epsilon\big\|_{L^2_T(L^2)} \leq C\big(\|u_{0,\epsilon}\|_{L^2}, \|b_{0,\epsilon}\|_{L^2}\big).
$$

This means that the derivatives of the difference functions $\nabla\delta u_\varepsilon$ and $\nabla\delta b_\varepsilon$ also have L^2 regularity and, thanks to the entropy inequality (5.42), they will enjoy similar bounds. Thus, for any small $\eta>0$, we can estimate

$$
\begin{aligned}
h(\epsilon)\left|\int_\Omega \mu(\rho_\epsilon)(\nabla\times B)(\nabla\times\delta b_\varepsilon)\,\mathrm{d}x\right| &\leq \sqrt{h(\epsilon)}\;\|\nabla\times B\|_{L^2}\big\|\sqrt{h(\epsilon)}\;\mu(\rho_\epsilon)\nabla\times\delta b_\varepsilon\big\|_{L^2} \\
&\leq \eta\big\|\sqrt{h(\epsilon)}\;\mu(\rho_\epsilon)\nabla\times\delta b_\varepsilon\big\|_{L^2_T(L^2)}^2 + h(\epsilon)C\big(T,\eta,\|(R_0,U_0,B_0)\|_{H^s}\big).
\end{aligned}
$$

Exactly in the same way, we also have

$$h(\epsilon)\left|\int_\Omega \nu(\rho_\epsilon)\nabla U : \nabla\delta u_\varepsilon \,\mathrm{d}x\right| \leq \eta\left\|\sqrt{h(\epsilon)}\;\nu(\rho_\epsilon)\nabla\delta u_\varepsilon\right\|^2_{L^2_T(L^2)} + h(\epsilon)C\big(T,\eta,\|(R_0,U_0,B_0)\|_{H^s}\big).$$

Only the last integral J_6 remains. It involves the time derivative $\partial_t U$, whose regularity we must ascertain. Because $(R,U,B)\in L^\infty_T(H^s)$ solves system (5.10), we have

$$\partial_t U = \mathbb{P}\left(\operatorname{div}\big(B\otimes B - U\otimes U\big) + \frac{1}{2}RU^\perp\right).$$

Now, since H^{s-1} is a Banach algebra (we have $s>2=1+d/2$), we deduce the estimate

$$\|\partial_t U\|_{L^\infty_T(H^{s-1})} \leq \|U\|^2_{L^\infty_T(H^s)} + \|B\|^2_{L^\infty_T(H^s)} + \|R\|^2_{L^\infty_T(H^s)}.$$

Using the embedding $H^s\hookrightarrow L^\infty$, we finally gather

$$\begin{aligned}
\epsilon\left|\int_\Omega r_\epsilon\big(\partial_t U + (u_\epsilon\cdot\nabla)U\big)\cdot\delta u_\varepsilon\,\mathrm{d}x\right| &\leq \epsilon\|\partial_t U\|_{L^\infty}\|r_\epsilon\|_{L^2}\|\delta u_\varepsilon\|_{L^2} + \epsilon\|u_\epsilon\|_{L^2}\|\nabla U\|_{L^\infty}\|\delta u_\varepsilon\|_{L^2}\\
&\leq \|\delta u_\varepsilon\|^2_{L^2} + \epsilon^2\|\partial_t U\|^2_{H^{s-1}}\|r_\epsilon\|^2_{L^2} + \epsilon^2\|U\|^2_{H^s}\|u_\epsilon\|^2_{L^2}\\
&\leq \|\delta u_\varepsilon\|^2_{L^2} + \epsilon^2 C\big(T,\|(R_0,U_0,B_0)\|_{H^s},\|(r_{0,\epsilon},u_{0,\epsilon},b_{0,\epsilon})\|_{L^2}\big).
\end{aligned}$$

Putting all the estimates for $J_1,\ldots,J_6$ together and choosing η small enough, we get

$$\begin{aligned}
\mathcal{E}\Big([r_\epsilon,u_\epsilon,b_\epsilon]\big|[R,U,B]\Big)(T) + \frac{1}{2}h(\epsilon)\int_0^T\int_\Omega\Big\{\nu(\rho_\epsilon)|\nabla\delta u_\varepsilon|^2 + \mu(\rho_\epsilon)|\nabla\delta b_\varepsilon|^2\Big\}\,\mathrm{d}x\,\mathrm{d}t&\\
\leq \int_0^T\mathcal{E}\Big([r_\epsilon,u_\epsilon,b_\epsilon]\big|[R,U,B]\Big)\,\mathrm{d}t + \big(h(\epsilon)+\epsilon^2\big)\,C&,
\end{aligned}$$

for a suitable constant $C = C\Big(T,\|(R_0,U_0,B_0)\|_{H^s},\|(r_{0,\epsilon},u_{0,\epsilon},b_{0,\epsilon})\|_{L^2}\Big) > 0$. An application of Grönwall's lemma gives estimate (5.11), completing in this way the proof to Theorem 5.7.

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Etude mathématique de fluides en interaction avec un champ magnétique

Résumé. Nous nous proposons, dans ce travail de thèse, d'explorer quelques spécificités de la magnétohydrodynamique incompressible. Il s'agit d'un système d'Équations aux Dérivées Partielles décrivant les fluides conducteurs de courant subissant l'effet des champs magnétiques qu'ils génèrent. Le modèle, conçu pour l'étude des plasmas, présente dans son analyse mathématique des difficultés singulières, que nous voulons aborder. Nous nous intéresserons en tout particulier à la résolution du problème de Cauchy dans des espaces de fonctions bornées et à l'étude du temps de vie des solutions. Notre cheminement passera par deux grandes étapes cruciales. Une première partie de notre étude portera sur les particularités des solutions bornées en mécanique des fluides incompressibles et l'utilisation de la projection de Leray dans ce cadre. Deuxièmement, nous mettrons en évidence le rôle des variables d'Elsässer dans la structure du système et la théorie des solutions dans les espaces de Besov. Le manuscrit aborde également d'autres questions liées aux espaces de Besov homogènes ou des problèmes de perturbations singulières.

Mots-clés : magnétohydrodynamique, fluides incompressibles, analyse de Littlewood-Paley, solutions bornées, perturbations singulières.

Mathematical Study of Fluids Interacting with a Magnetic Field

Abstract. In this dissertation, we will explore some of the features of incompressible magnetohydrodynamics. More specifically, it is a system of Partial Differential Equations originating in plasma physics and describing the dynamics of conducting fluids under the influence of a self-generated magnetic field. The mathematical study of the model introduces a series of challenging questions. In particular, we will focus on solving the initial value problem in spaces of bounded functions and finding estimates for the lifespan of solutions. Our discussion will have two main steps. Firstly, the specific features of bounded solutions in incompressible hydrodynamics will necessitate a thorough analysis, as well as the use of the Leray projection in that framework. Secondly, we will highlight the role of the Elsässer variables in the structure of the system and the theory of solutions in Besov spaces. Other aspects of the manuscript include issues linked to homogeneous Besov spaces and singular perturbation problems.

Keywords: magnetohydrodynamics, incompressible fluids, Littlewood-Paley analysis, bounded solutions, singular perturbation problems.

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