2. Theorem: Let X be a smooth projective variety and $D = \sum d_i D_i$ be a nc $\mathbb{Q}$-divisor with $0 \le d_i \le 1$. If A is an ample $\mathbb{Q}$-divisor with $D+A$ a $\mathbb{Z}$-divisor, then

$$H^p(X, K_X + D + A) = 0 \quad \text{for } p > 0$$

Remark: This is the log version of Kodaira vanishing for the log pair (X,D), for $K_X + D$ is the log canonical bundle of (X,D)

The following assertions are equivalent to the theorem.

2a. X, D, A as before, but $0 \le d_i < 1$.
Then $H^p(X, K_X + D + A) = 0$ for $p > 0$

(2. $\Rightarrow$ 2a. is clear
2a. $\Rightarrow$ 2. Apply 2a to $D' := (1-\varepsilon) D$, $A' := A + \varepsilon D$ for $0 < \varepsilon << 1$. Then use that the ample cone is open and that therefore A' is still ample.)

2b. Suppose X smooth projective, A ample $\mathbb{Q}$-divisor with $\ulcorner A \urcorner - A$ nc. Then $H^p(X, K_X + \ulcorner A \urcorner) = 0$ for $p > 0$

(2a $\Leftrightarrow$ 2b : $D := \ulcorner A \urcorner - A$)

<u>Proof</u> of 2a) for the case $D = d_1 D_1$
(The general case is only more difficult in the notation.)

<u>1st step</u> Write $d_1 = k/m$ with $0 \le k < m$

Assume $\mathcal{O}(D_1) = L^m$ for some $m \in \operatorname{Pic}(X)$.
Then take the m-th root of $0 \neq s \in H^0(X, L^m)$:
$\pi: X' \to X$. In particular, $\pi_* \mathcal{O}_{X'} = \mathcal{O}_X \oplus L^* \oplus \ldots \oplus L^{1-m}$

Note that X' is smooth, for D_1 is nc.

Since $A + d_1 D_1$ is $\mathbb{Z}$-divisor, also $\pi^*(A + d_1 D_1)$ is.
But $\pi^* d_1 D_1 = d_1 (m L) = k L$ and therefore $\pi^* A$ is a $\mathbb{Z}$-divisor. Moreover, since π is finite, also $\pi^* A$ is ample. Thus, by Kodaira vanishing

$$H^p(X', -\pi^* A) = 0 \quad \text{for} \quad p < \dim(X') = \dim(X)$$

Projection formula $\Rightarrow H^p(X, -A) \oplus \ldots \oplus H^p(X, L^{1-m}(-A)) = 0$.

In particular $H^p(X, L^{-k}(-A)) = 0$, but writing $L^k(A) = A + \frac{k}{m} D_1$ then yields the assertion.

<u>2nd step</u>. As before $d_1 = k/m$ with $0 \le k < m$
Then take m-th root à la Bloch-Gieseker:
$\pi: X' \to X$ with X' smooth, $\pi^* D_1$ nc and
$\pi^* \mathcal{O}(D_1) = L^m$ for some $L \in \operatorname{Pic}(X')$

By step 1 and using that $\mathcal{O}_X$ is a direct summand of $\pi_* \mathcal{O}_{X'}$ one finds the assertion. $\square$

3. Theorem: Let X be smooth projective, $D = \sum d_i D_i$ a snc divisor with $0 \le d_i < 1$, A nef + big $\mathbb{Q}$-divisor s.t. $A + D$ is $\mathbb{Z}$-divisor.

Then $H^p(X, K_X + D + A) = 0$ for $p > 0$.

This is a combination of Thm. 1 & 2. Note that in this situation one has to assume $d_i < 1$, as the cone of nef & big divisors is not open.

We leave this as an exercise.

Consequences:

1) The Grauert - Riemenschneider vanishing

Suppose $\pi : X \to Y$ is a generically finite, projective morphism and X smooth. (X, Y projective)

Then $R^p \pi_* K_X = 0$ for $p > 0$

(Earlier, we had a similar statement for resolutions obtained as sequences of small blow-ups.)

Proof. Let $L \in \mathrm{Pic}(Y)$ be ample. Then $\pi^* L$ is nef and big.

Thus, $H^p(X, K_X + \pi^* L^m) = 0$ for $p > 0$ and all $m > 0$.

Now use the Leray SS

$$E_2^{pq} = \underbrace{H^p(Y, (R^q \pi_* K_X) \otimes L^m)}_{} \Rightarrow H^{p+q}(X, K_X \otimes \pi^* L^m)$$

$= 0$ for $p > 0$ and $m \gg 0$

(Serre vanishing)

Hence $H^0(Y, (R^q\pi_* K_X) \otimes L^m) = H^q(X, K_X \otimes \pi^* L^m) = 0$

for $q > 0$, $m \gg 0$. Therefore, $R^q \pi_* K_X = 0$ for $q > 0$. $\square$

ii) One similarly proves the relative version of Theorem 2 (resp. 3)

Let $\pi: X \to Y$ be a projective morphism, X smooth, A π-ample with $\ulcorner A \urcorner - A$ nc. Then

$$R^p \pi_* (K_X + \ulcorner A \urcorner) = 0 \quad \text{for } p > 0$$

Proof (Under the additional assumption: Y proj.)

Pick ample $\mathbb{Z}$-divisor H on Y and consider Leray SS

$$E_2^{qp} = H^q(Y, R^p\pi_*(K_X + \ulcorner A \urcorner + m\pi^* H)) \Rightarrow H^{p+q}(X, K_X + \ulcorner A \urcorner + m\pi^* H)$$

$= 0$ for $q > 0$ and $m \gg 0$

(by Serre vanishing)

Hence, for $m \gg 0$

$$H^0(Y, R^p\pi_*(K_X + \ulcorner A \urcorner + m\pi^* H)) = H^p(X, K_X + \ulcorner A \urcorner + m\pi^* H)$$

$= H^0(Y, R^p\pi_*(K_X + \ulcorner A \urcorner) + mH)$

$= 0$ { for $m \gg 0$, by Thm 2, as $A + m\pi^* H$ ample $m \gg 0$ and $\ulcorner A + m\pi^* H \urcorner - (A + m\pi^* H) = \ulcorner A \urcorner - A$ still nc

$\Rightarrow H^0(Y, R^p\pi_*(K_X + \ulcorner A \urcorner) + mH) = 0$ for $m \gg 0$

$\Rightarrow R^p\pi_*(K_X + \ulcorner A \urcorner) = 0$ $\square$

Note : A π-nef and π-big is enough.

4. Theorem (Singular Kawamata-Viehweg vanishing)

Let $(X, D = \sum d_i D_i)$ be a log pair (i.e. X normal, $0 \le d_i \le 1$).

(*) If (X, D) is weakly log terminal and A is ample,

then $H^p(X, K_X + A + D) = 0$ for $p > 0$, if

$K_X + A + D$ is an integral Cartier divisor.

Remarks: i) (*) could be replaced by

($\widetilde{*}$) If (X, D) weakly Kawamata log terminal and

A big and nef. (In particular, $0 \le d_i < 1$)

ii) The "weakly" in both cases means that in addition to the usual definition the resolution

$f : Y \to X$ can be chosen such that there

exists an f-ample divisor $-\sum \delta_i E_i$ with $0 < \delta_i \ll 1$

and E_i are all the exceptional divisors.

(If X is $\mathbb{Q}$-factorial this is no extra condition

$\to$ [Matsuki, p. 176])

Proof: Since (X, D) is log terminal, there exists a resolution $f: Y \to X$ s.t. $D_Y := \hat{D} + \sum E_i$ is nc and $K_Y + D_Y = f^*(K_X + D) + \sum b_i E_i$ with $b_i > 0$

Moreover, we assumed that there exists a linear combination $-\sum \delta_i E_i$ (with $0 < \delta_i \ll 1$) that is f-ample.

Then (Y, D_Y) with the ample divisor $A_Y := f^*A - \sum \delta_i E_i$ satisfies the assumption of Theorem 2.

Thus $H^p(Y, K_Y + D_Y + A_Y) = 0$ for $p > 0$ if $K_Y + D_Y + A_Y$ is integral. Need slight modification: Replace D_Y by $D_Y' := D_Y - \sum e_i E_i$ with $e_i = \{ b_i - \delta_i \}$ "fractional part".

Since by assumption $K_X + D + A$ is integral, also $f^*(K_X + D + A)$ integral and hence also

$$f^*(K_X + D + A) + \sum (b_i - \delta_i - e_i) E_i = K_Y + D_Y' + A_Y.$$

We still have D_Y' effective and nc.

Now Thm 2 really applies and proves

$$H^p(Y, K_Y + D_Y' + A_Y) = 0 \quad \text{for } p > 0 \,!$$

In order to apply Leray spectral sequence, need to prove $R^i f_*(K_Y + D_Y' + A_Y) = 0$ for $i > 0$. (*)

$$\text{Then } H^p(Y, K_Y + D_Y' + A_Y) = H^p(X, f_*(K_Y + D_Y' + A_Y))$$

$$= H^p(X, K_X + D + A + \underbrace{f_* \mathcal{O}(\sum \underbrace{(b_i - \delta_i - e_i)}_{\geq 0} E_i)}_{\simeq \mathcal{O}})$$

$$= H^p(X, K_X + D + A)$$

In order to ensure (*) we need to find an f-ample divisor B with:
- $\lceil B \rceil - B \sim C$
- $\lceil B \rceil = D_Y' + A_Y$

Write $D_Y' + A_Y = \widehat{D} + f^* A + \sum (1 - e_i - \delta_i) E_i$

Use $0 < e_i + \delta_i \leq 1 \rightsquigarrow 0 \leq 1 - e_i - \delta_i < 1$

Then one finds $1 - e_i - \delta_i < c_i < 1$

with $\sum (1 - e_i - \delta_i - c_i) E_i$ f-ample and if

we set $B = D_Y' + A_Y - \sum c_i E_i$, then

$\lceil B \rceil = D_Y' + A_Y$ and B f-ample

Question: What is wrong in the above proof?

Answer: There are possibly exceptional divisors in $\widehat{D} = f^* D - \sum a_i E_i$ $(a_i \geq 0)$ that may destroy the f-ampleness!

The above proof was taken from Matsuki. I can't see how to repair it.

Here is the proof from [KMM]:

First, choose $0 < \delta \ll \min\{\delta_i\}$ such that $f^* A + \delta \widehat{D} - \sum \delta_i E_i$ is f-ample.

Then the ramification formula
$$K_Y + D_Y = f^*(K_X + D) + \sum b_i E_i \qquad b_i > 0$$

becomes
$$K_Y + \delta \widehat{D} = f^*(K_X + D) + \underbrace{\sum b_i E_i - \sum E_i - (\widehat{D} - \delta \widehat{D})}_{=: E}$$

- Study $\lceil E \rceil$: $\quad \cdot\ b_i > 0 \Rightarrow b_i - 1 > -1 \Rightarrow \lceil b_i - 1 \rceil \geq 0$

$\cdot\ 0 \leq d_i \leq 1 \Rightarrow 0 \geq (\delta - 1) d_i > -1$

$\Rightarrow \lceil E \rceil = \sum \lceil b_i - 1 \rceil E_i \geq 0$

$f_* \lceil E \rceil = \mathcal{O}$

- Now replace $K_Y + D_Y' + A_Y$ in Matsuki's proof by

$K_Y + \lceil \delta \hat{D} + f^* A \rceil = \lceil f^*(K_X + D + A) + E \rceil$

(Note $K_Y + \lceil \delta \hat{D} + f^* A \rceil = K_Y + \lceil \delta \hat{D} + f^* A - \sum \delta_i E_i \rceil$.)

<u>Check</u>: i) $R^i f_* (K_Y + \lceil \delta \hat{D} + f^* A \rceil) = 0 \qquad i > 0$

ii) $f_* (K_Y + \lceil \delta \hat{D} + f^* A \rceil) = K_X + D + A$

<u>Pf</u>: $\cdot\ R^i f_* (K_Y + \lceil \delta \hat{D} + f^* A \rceil) = R^i f_* (K_Y + \lceil \underbrace{\delta \hat{D} + f^* A - \sum \delta_i E_i}_{f\text{-ample with snc fractional part}} \rceil)$

$= 0$ (relative version of Thm 2)

$\cdot\ f_* (K_Y + \lceil \delta \hat{D} + f^* A \rceil) = f_* \lceil f^*(K_X + D + A) + E \rceil$

$= \underbrace{K_X + D + A}_{\text{integral}} + \underbrace{f_* \lceil E \rceil}_{= \mathcal{O}}$ ✓

Then $H^p(X, K_X + D + A) =$

$\underset{ii)}{=} H^p(X, f_*(K_Y + \lceil \delta \hat{D} + f^* A \rceil)) \underset{i)}{=} H^p(Y, K_Y + \lceil \delta \hat{D} + f^* A \rceil)$

$= H^p(Y, K_Y + \lceil \delta \hat{D} + f^* A - \sum \delta_i E_i \rceil) \underset{\text{Thm 2}}{=} 0$ $\square$