Submodels of Prikry Generic Extensions

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Theorem 1. (Gitik, Kanovei, K.) *Let $M_0 \vDash U_0$ is a measure on κ_0 . Let C be a Prikry sequence for U_0 over M_0 . Then*

$$\forall Z \in M_0[C] \, \exists C' \subseteq C, C' \in M_0[C] \;\; M_0[Z] = M_0[C'].$$

This implies:

- *the constructibility degrees of sets in the Prikry generic extension over the ground model M_0 are parametrized by $\mathcal{P}(\omega)/\mathrm{fin}$.*

- *every proper intermediate inner model N, $M_0 \subsetneq N \subseteq M_0[C]$, of* ZFC *is a Prikry generic extension of the form $M_0[C']$ for some $C' \subseteq C$; hence Prikry forcing is a "minimal" forcing for singularizing a measurable cardinal.*

- $(\{N \mid M_0 \subseteq N \subseteq M_0[C]\}, \subseteq) \cong (\mathcal{P}(C)^{M_0[C]}, \subseteq/\mathrm{fin}) \cong (\mathcal{P}(\omega)^{M_0}, \subseteq/\mathrm{fin})$

We sketch the theorem for $Z \subseteq \kappa_0$ and for $Z \subseteq \kappa_0^+$.

1. Prikry forcing

Definition 2. ***Prikry forcing*** *is the partial order* $(P, \leqslant)$ *defined by*

$$P = \{(a, A) | a \in [\kappa_0]^{<\omega}, A \in U_0, \max(a) < \min(A)\}$$

and

$$(a, A) \leqslant (b, B) \text{ iff } a \setminus b \subseteq B \wedge A \subseteq B.$$

If G is P-generic over M_0 then

$$C = \bigcup_{(a, A) \in G} a$$

is a **Prikry sequence** for U_0, i.e.

$$\forall A \in \mathcal{P}(\kappa_0) \cap M_0 \ (A \in U_0 \leftrightarrow C \setminus A \text{ is finite}).$$

Proposition 3.

a) $M_0[G] = M_0[C]$, *with* $G = \{(a, A) \in P | C \setminus A = a\}$

b) $V_{\kappa_0} \cap M_0 = V_{\kappa_0} \cap M_0[C]$.

c) *Cardinals are absolute between* M_0 *and* $M_0[C]$.

d) C *is cofinal in* κ_0 *of ordertype* ω.

Theorem 4. (A.Dodd, R.B.Jensen) *If a regular cardinal* κ *is turned into a singular cardinal of cofinality* ω *then* κ *is measurable in an inner model and there is a Prikry sequence for that measure.*

2. Iterated Ultrapowers

Definition 5. *Define the* ***iteration***

$$(M_m, U_m, \kappa_m, \pi_{m\,n})_{m \leqslant n \leqslant \omega}$$

of (M_0, U_0) *by recursion:*

- $\pi_{0\,0} = \mathrm{id}$
- $\pi_{m,m+1}\colon M_m \rightarrow M_{m+1} = \mathrm{Ult}(M_m, U_m)$ *is the ultrapower of* M_m *by* U_m
- $\pi_{i,m+1} = \begin{cases} \pi_{m,m+1} \circ \pi_{i\,m} \text{ if } i \leqslant m \\ \mathrm{id} \text{ if } i = m+1 \end{cases}$
- $U_{m+1} = \pi_{m,m+1}(U_m)$, $\kappa_{m+1} = \pi_{m,m+1}(\kappa_m)$
- M_ω, $(\pi_{m\,\omega})_{m \leqslant \omega}$ *is the* ***transitive*** *direct limit of the system* $(M_m, \pi_{m\,n})_{m \leqslant n < \omega}$
- $U_\omega = \pi_{0\,\omega}(U_0)$, $\kappa_\omega = \pi_{0\,\omega}(\kappa_0)$

Proposition 6.

a) $\pi_{m\omega} \upharpoonright \kappa_m = \mathrm{id}$

b) $M_m = \{\pi_{0m}(f)(\kappa_0, \ldots, \kappa_{m-1}) \mid f \in M_0\,,\, f\colon \kappa_0^m \to M_0\}$

c) $\forall A \in \mathcal{P}(\kappa_\omega) \cap M_\omega$ $(A \in U_\omega \leftrightarrow \{\kappa_m | m < \omega\} \setminus A$ *is finite*$)$, *i.e.*, $\{\kappa_m | m < \omega\}$ *is a Prikry sequence for* U_ω.

3. An Intersection Model

Set $M = M_\omega$, $\kappa = \kappa_\omega$, $U = U_\omega$, $D = \{\kappa_m | m < \omega\}$.

Definition 7. *Define an* ***intersection model*** *by*

$$N = \bigcap_{m<\omega} M_m$$

Proposition 8. *The intersection model N equals $M[D]$, the Prikry extension of M by D.*

Theorem 9.

$$\forall Z \subseteq \kappa, Z \in M[D] \, \exists D' \subseteq D \; M[Z] = M[D']$$

Wellorder ascending sequences $\alpha_0 < ... < \alpha_{m-1}$ and $\beta_0 < ... < \beta_{n-1}$ lexicographically from the top: $(\alpha_0, ..., \alpha_{m-1}) \prec (\beta_0, ..., \beta_{n-1})$ iff there is some i such that

$\alpha_{m-1} = \beta_{n-1}$, ..., $\alpha_{m-i} = \beta_{n-i}$, β_{n-i-1} exists, and if α_{n-i-1} exists, then $\alpha_{m-i-1} < \beta_{n-i-1}$.

Lemma 10. *Let* $u \in M_n$. *Let* $\alpha_0 < \ldots < \alpha_{m-1}$ *be* $\prec$ *-minimal such that there is* $f \in M_0$, $f : \kappa_0^m \rightarrow M_0$ *such that*

$$u = \pi_{0n}(f)(\alpha_0, \ldots, \alpha_{m-1}).$$

Then $\{\alpha_0, \ldots, \alpha_{m-1}\} \subseteq \{\kappa_0, \ldots, \kappa_{n-1}\}$.

If $\alpha_0 < \ldots < \alpha_{m-1}$ *is* $\prec$ *-minimal such that*

$$u = \pi_{0n}(f)(\alpha_0, \ldots, \alpha_{m-1})$$

and if moreover $u \subseteq \kappa_n$ *then* $\alpha_0 < \ldots < \alpha_{m-1}$ *is* $\prec$ *-minimal such that*

$$u = \pi_{0\omega}(f)(\alpha_0, \ldots, \alpha_{m-1}) \cap \kappa_n .$$

Proof. Assume that $\{\alpha_0, \ldots, \alpha_{m-1}\} \nsubseteq \{\kappa_0, \ldots, \kappa_{n-1}\}$ and let i be maximal such that $\alpha_i \notin \{\kappa_0, \ldots, \kappa_{n-1}\}$. Let κ_l be minimal such that $\alpha_i < \kappa_l$. By the representation theorem there is some $g \in M_0$, $g\colon \kappa_0^l \to M_0$ such that

$$\alpha_i = \pi_{0l}(g)(\kappa_0, \ldots, \kappa_{l-1}).$$

Then

$$\alpha_i = \pi_{0n}(g)(\kappa_0, \ldots, \kappa_{l-1}).$$

Let $\beta_0 < \ldots < \beta_{r-1}$ enumerate

$$\{\kappa_0, \ldots, \kappa_{l-1}\} \cup \{\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{m-1}\}.$$

Note that $(\beta_0, \ldots, \beta_{r-1}) \prec (\alpha_0, \ldots, \alpha_{m-1})$.

Let

$$(\kappa_0, \ldots, \kappa_{l-1}) = (\beta_{j_0}, \ldots, \beta_{j_{l-1}})$$

and

$$(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{m-1}) = (\beta_{k_0}, \ldots, \beta_{k_{i-1}}, \beta_{k_{i+1}}, \ldots, \beta_{k_{m-1}}).$$

Define $h\colon \kappa_0^r \to M_0$ by

$$h(\xi_0, \ldots, \xi_{r-1}) = f(\xi_{k_0}, \ldots, \xi_{k_{i-1}}, g(\xi_{j_0}, \ldots, \xi_{j_{l-1}}), \xi_{k_{i+1}}, \ldots, \xi_{k_{m-1}}).$$

Then

$$\begin{aligned} u &= \pi_{0n}(f)(\alpha_0, \ldots, \alpha_{m-1}) \\ &= \pi_{0n}(f)(\alpha_0, \ldots, \alpha_{i-1}, \pi_{0n}(g)(\kappa_0, \ldots, \kappa_{l-1}), \alpha_{i+1}, \ldots, \alpha_{m-1}) \\ &= \pi_{0n}(f)(\beta_{k_0}, \ldots, \beta_{k_{i-1}}, \pi_{0n}(g)(\beta_{j_0}, \ldots, \beta_{j_{l-1}}), \beta_{k_{i+1}}, \ldots, \beta_{k_{m-1}}) \\ &= \pi_{0n}(f)(\beta_0, \ldots, \beta_{r-1}) \end{aligned}$$

contradicting the minimality of $(\alpha_0, \ldots, \alpha_{m-1})$. □

Proof of Theorem 9.

For $Z \in M$ the theorem is obvious. So consider $Z \subseteq \kappa$, $Z \in M[D] \setminus M$.

Lemma 11. *κ is singular in $M[Z]$.*

Proof. Assume not. For $m < \omega$ let

$$Z = \pi_{0m}(f_m)(\kappa_0, \ldots, \kappa_{m-1}) \in M_m\,.$$

Then $Z \cap \kappa_m = \pi_{0m}(f_m)(\kappa_0, \ldots, \kappa_{m-1}) \cap \kappa_m$ and

$$Z \cap \kappa_m = \pi_{0\omega}(f_m)(\kappa_0, \ldots, \kappa_{m-1}) \cap \kappa_m\,.$$

So in the model $M[Z]$,

$$\forall \zeta < \kappa \exists m < \omega \exists \xi_0, \ldots, \xi_{m-1} < \zeta \colon Z \cap \zeta = \pi_{0\omega}(f_m)(\xi_0, \ldots, \xi_{m-1}) \cap \zeta.$$

This defines **regressive** functions, and there are values m_0 and η_0, ..., η_{m_0} such that for a stationary set $S \subseteq \kappa$

$$\forall \zeta \in S\; Z \cap \zeta = \pi_{0\omega}(f_{m_0})(\eta_0, \ldots, \eta_{m_0-1}) \cap \zeta.$$

But then

$$Z = \pi_{0\omega}(f_{m_0})(\eta_0, \ldots, \eta_{m_0-1}) \in M.$$

Contradiction. $\square$

Lemma 12. *In $M[Z]$, there is an infinite subset $D_0 \subseteq D$ (which is cofinal in κ).*

Proof. Let $\{\alpha_\nu | \nu < \gamma\} \in M[Z]$ be cofinal in κ where $\gamma < \kappa$. Without loss of generality, $\gamma < \kappa_0$.
Work in M_0. For $\nu < \gamma$ consider the minimal κ_m such that $\alpha_\nu < \kappa_m$ and a $\prec$-minimal sequence $\vec{\kappa}_\nu \subseteq D$ such that for some f_ν

$$\alpha_\nu = \pi_{0m}(f_\nu)(\vec{\kappa}_\nu).$$

Since $\gamma < \kappa_0$

$$(\pi_{0\omega}(f_\nu) | \nu < \gamma) = \pi_{0\omega}((f_\nu | \nu < \gamma)) \in M$$

we can, in $M[Z]$, define $\vec{\kappa}_\nu$ as the $\prec$-minimal sequence such that

$$\alpha_\nu = \pi_{0\omega}(f_\nu)(\vec{\kappa}_\nu).$$

Let $D_0 = \bigcup_{\nu < \gamma} \vec{\kappa}_\nu \in M[Z]$, $D_0 \subseteq D$. If D_0 were finite then

$$\{\alpha_\nu | \nu < \gamma\} \subseteq \{\pi_{0\omega}(f_\nu)(\vec{\kappa}) | \nu < \gamma, \vec{\kappa} \subseteq D_0\} \in M$$

would make κ singular in M, contradiction. $\square$

Work in M_0. Let $\lambda_0 < \lambda_1 < \ldots$ enumerate D_0. For $m < \omega$ let $\vec{\kappa}_m \subseteq D$ be $\prec$-minimal such that there is $f_m \in M_0$, $f_m: \kappa_0^{\text{length}(\vec{\kappa}_m)} \to M_0$ such that

$$Z \cap \lambda_m = \pi_{0\omega}(f_m)(\vec{\kappa}_m) \cap \lambda_m. \tag{1}$$

Let $D' = D_0 \cup \bigcup_{m<\omega} \vec{\kappa}_m \subseteq D$. Observe that

$$(\pi_{0\omega}(f_m)|m < \omega) = \pi_{0\omega}((f_m|m < \omega)) \in M. \tag{2}$$

By (1) and (2), $Z \in M[D']$.

Conversely, $D_0 \in M[Z]$, and $(\vec{\kappa}_m|m < \omega)$ can be defined in $M[Z]$ by: $\vec{\kappa}_m$ is $\prec$-minimal such that

$$Z \cap \lambda_m = \pi_{0\omega}(f_m)(\vec{\kappa}_m) \cap \lambda_m.$$

Hence $D' \in M[Z]$.

Thus $M[Z] = M[D']$. $\square$

Proof of Theorem 1 for $Z \subseteq \kappa_0$.
We want to show that the top condition

$$(\emptyset, \kappa_0) \Vdash \Phi(\dot{C}) \equiv \forall Z \subseteq \kappa_0 \, \exists C' \subseteq \dot{C} \; M_0[Z] = M_0[C'],$$

Assume not, and let $M_0 \vDash$ "$(a, A) \Vdash \neg\Phi(\dot{C})$".
By elementarity, $M \vDash$ "$(\pi_{0\omega}(a), \pi_{0\omega}(A)) \Vdash \neg\Phi(\dot{C})$".
Let $\{\kappa_m | n \leqslant m < \omega\} \subseteq \pi_{0\omega}(A)$. Then $\pi_{0\omega}(a) \cup \{\kappa_m | n \leqslant m < \omega\}$ is a Prikry sequence for $\pi_{0\omega}(U_0)$ and

$$M[\pi_{0\omega}(a) \cup \{\kappa_m | n \leqslant m < \omega\}]$$

is a generic extension where $(\pi_{0\omega}(a),\ \pi_{0\omega}(A))$ is in the generic filter corresponding to $\pi_{0\omega}(a) \cup \{\kappa_m | n \leqslant m < \omega\}$. Hence

$$M[\pi_{0\omega}(a) \cup \{\kappa_m | n \leqslant m < \omega\}] \vDash \neg\Phi(\pi_{0\omega}(a) \cup \{\kappa_m | n \leqslant m < \omega\})$$

Since the model $M[C]$ and the formula $\Phi(C)$ are invariant w.r.t. finite variations of C

$$M[D] \vDash \neg\Phi(D)$$

But this contradicts Theorem 9. $\square$

4. Dense Projections and two-stage iterations

Definition 13. *Let* $(P, \leqslant)$, $(P', \leqslant)$ *be partial orders.* $\pi: P \rightarrow P'$ *is a* ***dense projection*** *if*

- $p \leqslant q \rightarrow \pi(p) \leqslant \pi(q)$
- $\pi[P]$ *is dense in* P'
- $\forall p' \leqslant \pi(p) \exists q \leqslant p\, p(q) \leqslant p'$

Theorem 14. *Let* $\pi: P \rightarrow P'$ *be a dense projection,* $p \in M$. *Let* G *be* M*-generic over* P*. Set*

$$G' = \{p' \in P' | \exists p \in G\, p' \geqslant \pi(p)\}.$$

Then

- G' *is* M*-generic over* P'*.*
- $\pi^{-1}[G'] \subseteq P$ *is a partial order,* $\pi^{-1}[G'] \in M[G']$*.*
- G *is* $M[G']$*-generic over* $\pi^{-1}[G']$*.*
- $M[G] = M[G'][G]$ *is a two-stage extension by* $P' * \pi^{-1}[\dot{G}']$*;* $\pi^{-1}[\dot{G}']$ *is the quotient* P/P'*.*
- $p' \Vdash_{P'} p \in \pi^{-1}[\dot{G}'] \Leftrightarrow p' \Vdash_{P'} \pi(p) \in \dot{G}' \Leftrightarrow p' \leqslant \pi(p)$*.*

Now let P be Prikry forcing over M with normal measure U. Define a dense projection $\pi\colon P \rightarrow P$

$$\pi(\{a_0, a_1, \ldots, a_{2n-1}\}, A) = (\{a_0, a_2, \ldots, a_{2n-2}\}, A')$$

and

$$\pi(\{a_0, a_1, \ldots, a_{2n}\}, A) = (\{a_0, a_2, \ldots, a_{2n}\}, A')$$

where $a_0 < a_1 < \ldots < a_{2n-1} < a_{2n} < \kappa$ and $A' = \{a \in A | a \text{ is a limit of } A\} \in U$.
Let $M[G]$ be a Prikry extension with Prikry sequence

$$C = \{c_0, c_1, \ldots\},\ c_0 < c_1 < \ldots$$

The projection G' of G is given by the Prikry sequence

$$\bigcup_{(a,A) \in G'} a = \bigcup_{(a,A) \in G'} \pi(a) = C_{\text{even}} = \{c_0, c_2, c_4 \ldots\}.$$

Hence $M[G] = M[C_{\text{even}}][G]$, G is $M[C_{\text{even}}]$-generic over $\pi^{-1}[G']$. Denote the quotient forcing by

$$P / \dot{C}_{\text{even}}$$

Theorem 15. *$M[G] \models P/C_{\text{even}}$ has the κ^+-chain condition.*

Proof. Consider $\dot{p}^G = (\dot{p}_\alpha^G | \alpha < \kappa^+) \in P/C_{\text{even}}$. We may assume that this is forced by the weakest condition and it suffices to find a condition which forces the compatibility of some $\dot{p}_\alpha$ and $\dot{p}_\beta$. For $\alpha < \kappa^+$ choose q_α , $p_\alpha \in P$ such that

$$q_\alpha \Vdash \dot{p}_\alpha = p_\alpha \in P/\dot{C}_{\text{even}}$$

We may assume that the stems of q_α and p_α have the same odd length. By a pigeon principle we may assume that for all $\alpha < \kappa^+$

$$\begin{aligned} q_\alpha &= (\{a_0, b_1, a_2, b_3, \ldots, a_{2n}\}, B_\alpha) \\ p_\alpha &= (\{a_0, a_1, a_2, a_3, \ldots, a_{2n}\}, A_\alpha) \end{aligned}$$

for fixed $a_0 < b_1 < a_2 < b_3 < \ldots < a_{2n}$ and $a_0 < a_1 < a_2 < a_3 < \ldots < a_{2n}$. Then

$$(\{a_0, b_1, a_2, b_3, \ldots, a_{2n}\}, (B_\alpha \cap B_\beta)') \Vdash p_\alpha, p_\beta \text{ are compatible in } P/\dot{C}_{\text{even}} .$$

□

Towards a Proof of Theorem 1 for subsets of κ^+.

Consider $Z \subseteq \kappa^+$, $Z \in M[C]$. Every $Z \cap \alpha$, $\alpha < \kappa^+$ is equivalent to some $C_\alpha \subseteq C$:

$$M[Z \cap \alpha] = M[C_\alpha].$$

By pigeon principle, there is a fixed C', say C_{even}, such that for every $\alpha < \kappa^+$

$$M[Z \cap \alpha] = M[C_{\text{even}}].$$

Z is $M[C]$-generic over P/C_{even}. Let $\dot{Z}$ be a P/C_{even}-name, $\dot{Z}^G = Z$. For every $\alpha < \kappa^+$ define the Boolean value

$$b_\alpha = \|\dot{Z} \cap \check{\alpha} = (Z \cap \alpha)\check{}\,\|.$$

$(b_\alpha | \alpha < \kappa^+)$ is decreasing in the complete Boolean algebra for P/C_{even}, and by the κ^+-c.c. it is eventually constant. Let b_* be that constant value. Then

$$Z = \bigcup \{z \in M[C_{\text{even}}] \,|\, \exists \alpha < \kappa^+ \, \|\dot{Z} \cap \check{\alpha} = \check{z}\,\| = b_*\} \in M[C_{\text{even}}].$$

Thank you!