Resume of my master's thesis: *Flexibility and symplectic fillability*

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Background and motivation My master's thesis concerns an interplay between flexibility and rigidity features in the context of symplectic fillings.

Flexibility relates to a fascinating dichotomy in symplectic geometry, going back to Gromov [Gro87], of flexibility and rigidity. In many situations, symplectic problems behave rigidly: the possible behaviour is restricted compared to e.g. the underlying complex geometry, usually because there is a symplectic invariant in the background which restricts the possible behaviours. There are, however, some settings which admit an h-principle: the behaviour is *only* governed by some algebro-topological data (which derives from a formal analogue of the symplectic structure). This means that all symplectic invariants vanish and the objects behave rather flexibly.

The symplectic filling problem concerns, roughly speaking, the following question: when is a given $(2n-1)$-dimensional contact manifold the boundary of a compact $2n$-dimensional symplectic manifold (its *filling*)? Clearly, a compatibility condition between the symplectic structure on the filling and the contact structure on the boundary is required. There are several useful conditions leading to different kinds of *symplectic fillings*, such as Weinstein fillings, Liouville fillings, strong and weak fillings. There is a hierarchy between these kinds of fillings: in order of strictness, we have weak $<$ strong $<$ Liouville $<$ Weinstein fillings. By now, all inclusions have been shown to be proper in every dimension.

Apart from the existence of fillings, one can ask whether fillings of a given contact manifold are unique (up to a suitable notion of equivalence). While there are contact manifolds in every dimension which admit infinitely many non-homotopic Liouville fillings, certain classes of contact manifolds exhibit rigidity phenomena: their filling is determined up to homeomorphism, diffeomorphism or even up to symplectic deformation.

Notably, such rigidity results are much weaker in higher dimensions than in dimension four. The prototypical result in higher dimensions is due to Eliashberg, Floer and McDuff [McD91, Theorem 1.5], determining the diffeomorphism type of (symplectically aspherical strong) fillings of standard contact spheres in all dimensions. Barth, Geiges and Zehmisch have generalised this to certain subcritically Stein fillable manifolds [BGZ16].

There are also many results classifying symplectic fillings in dimension four up to deformation equivalence. These are notably absent in higher dimensions; the closest result until recently was due to Ivan Smith [Sei08, Corollary 6.5], proving that fillings of standard contact spheres have vanishing symplectic homology. The vanishing of symplectic homology lends support to the conjecture that the symplectic fillings should be unique. Zhengyi Zhou [Zho18], based on work by Oleg Lazarev [Laz17], extended Smith' result to fillings of flexible Weinstein fillable contact manifolds, achieving a new milestone for higher-dimensional symplectic topology.

My thesis was devoted to presenting Zhou's result, together with its necessary background, in detail. As such, it did not require original research; yet, I have been told that my account has been rather helpful to new students in my working group and beyond.

Precise result and sketch of argument The precise statement of Zhou's result is the following. We call a Liouville filling W of M *topologically simple* iff $c_1(W) = 0$ and the map $\pi_1(M) \to \pi_1(W)$ induced by the inclusion is injective.

Theorem 1 ([Zho18])**.** *Suppose an asymptotically dynamically convex contact manifold (M, ξ) with $\dim M \geq 5$ admits a topologically simple Liouville filling W with $SH(W) = 0$. Then every topologically simple Liouville filling W' of M satisfies $SH^*(W'; \mathbb{Z}) = 0$ and $H^*(W; \mathbb{Z}) \cong H^*(W'; \mathbb{Z})$.*

Zhou's argument has two main components. The first part is showing that the positive symplectic homology $SH^+(W')$ is independent of the filling W' of M. Since the positive symplectic homology is generated by the Reeb orbits in M, its generators are independent of the filling W'. The differential, however, counts Floer cylinders connecting such orbits, which can a priori enter the filling. One key obstacle is to exclude this possibility.

The first such result is due to Bourgeois and Oancea [BO09; BO16]. If all Reeb orbits in the Liouville fillable manifold M which are contractible in W' have positive (SFT) degree (such M is called *dynamically convex*), they showed independence of $SH^+(W')$ on W' using a neck-stretching argument: in summary, if a Floer cylinder were to enter the filling W', neck-stretching would produce a punctured Floer cylinder in the symplectisation of M, capped off by rigid holomorphic discs asymptotic to contractible Reeb orbits. Assuming appropriate transversality results, these orbits have degree zero, which contradicts the dynamical convexity hypothesis.

Zhou's proof uses a slightly different condition. The second part of Zhou's argument requires a class of manifolds which is closed under subcritical surgery, which dynamically convex manifolds are not. However, Oleg Lazarev [Laz17] found a suitably modified concept, called *asymptotically dynamically convex* (ADC) contact manifolds, and showed that ADC contact manifolds are closed under subcritical surgery.

In the second half of his argument, Zhou uses the boundary connected sum which is a particular kind of subcritical surgery. Using the vanishing of $SH(W)$ and the independence of $SH^+(W')$ of the filling W', the exact sequence of symplectic homology readily implies that $SH^+(W')$ has rank at most one. Forming the boundary connected sum $W' \natural W'$ yields a filling of the contact connected sum $M \# M$, which is again ADC, hence $SH^+(W' \natural W')$ satisfies the same rank bound. Since subcritical surgery induces an isomorphism on symplectic homology, vanishing of $SH^+(W')$ and $SH(W')$ follows suit.

Compared to Liouville domains, Weinstein domains have additional structure, which implies a handlebody decomposition. Flexible Weinstein domains are, up to Weinstein homotopy, characterised by their critical handles being attached along *loose Legendrians*. This implies an h-principle, and can be used to show that their symplectic homology vanishes. The handlebody decomposition also implies that a Weinstein filling is topologically simple. One can also show that flexible Weinstein domains are invariant under subcritical surgery. Hence, one obtains the following corollary.

Corollary 2 ([Zho18])**.** *Let (M, ξ) be a flexible Weinstein fillable ADC contact manifold with $\dim M \geq 5$ and $c_1(\xi) = 0$. Any topologically simple Liouville filling W of M satisfies $SH^*(W; \mathbb{Z}) = 0$.*

References

[BGZ16] Kilian Barth, Hansjörg Geiges, and Kai Zehmisch. “The diffeomorphism type of symplectic fillings”. arXiv:1607.03310. July 12, 2016.

[BO09] Frédéric Bourgeois and Alexandru Oancea. “An exact sequence for contact- and symplectic homology”. In: *Invent. Math.* 175.3 (2009), pp. 611–680.

[BO16] Frédéric Bourgeois and Alexandru Oancea. “S1-Equivariant Symplectic Homology and Linearized Contact Homology”. In: *International Mathematics Research Notices* 2017.13 (June 2016), pp. 3849–3937.

[Gro87] M. Gromov. “Soft and hard symplectic geometry”. In: *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986).* Amer. Math. Soc., Providence, RI, 1987, pp. 81–98.

[Laz17] Oleg Lazarev. “Contact manifolds with flexible fillings”. arXiv:1610.04837v2. Sept. 7, 2017.

[McD91] Dusa McDuff. “Symplectic manifolds with contact type boundaries.” In: *Invent. Math.* 103.3 (1991), pp. 651–671.

[Sei08] Paul Seidel. “A biased view of symplectic cohomology.” In: *Current developments in mathematics, 2006*. Somerville, MA: International Press, 2008, pp. 211–253.

[Zho18] Zhengyi Zhou. “Vanishing of Symplectic Homology and Obstruction to Flexible Fillability”. In: *International Mathematics Research Notices* (Nov. 2018).