Piecewise hereditary algebras and posets

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Motivation – piecewise hereditary posets

An abelian category $\mathcal{A}$ is *piecewise hereditary* if

$$\mathcal{D}^b(\mathcal{A}) \simeq \mathcal{D}^b(\mathcal{H})$$

for some hereditary abelian category $\mathcal{H}$ ($\operatorname{gl.dim} \mathcal{H} = 1$).
[Happel-Reiten-Smalø]

X – *poset* (finite partially ordered set)

k – field, kX – the *incidence algebra* of X over k

kX is *piecewise hereditary* if $\operatorname{mod} kX$ is piecewise hereditary.

Question. What distinguishes piecewise hereditary posets?

Several types of restrictions

- **Bounds on the global dimension**

  in terms of *connectivity properties* of the *graph of indecomposables* for finite length, piecewise hereditary categories.

- **Positivity properties of the Euler form**

  when the *Coxeter transformation* is periodic.

- **Weight types of canonical algebras**

  derived equivalent to incidence algebras.

The graph of indecomposables

$\mathcal{A}$ – finite length abelian category.

$\operatorname{ind}\mathcal{A}$ – isomorphism classes of indecomposables of $\mathcal{A}$.

$G(\mathcal{A})$ – *graph of indecomposables* of $\mathcal{A}$:

- vertices: the elements of $\operatorname{ind}\mathcal{A}$.
- edges: $Q \to Q'$ if $\operatorname{Hom}_{\mathcal{A}}(Q, Q') \neq 0$.

Let $k \geq 1$, $\varepsilon = (\varepsilon_0, \dots, \varepsilon_{k-1})$ – a sequence in $\{+1, -1\}^k$.

Let $Q, Q' \in \operatorname{ind}\mathcal{A}$. An *$\varepsilon$-path* from Q to Q' is a sequence of vertices

$$Q = Q_0, Q_1, \dots, Q_k = Q'$$

such that $Q_i \to Q_{i+1}$ if $\varepsilon_i = +1$ and $Q_{i+1} \to Q_i$ if $\varepsilon_i = -1$.

Bounds on the global dimension

$\mathcal{A}$ – finite length, *piecewise hereditary* category.

Theorem. If there exist $k \geq 1$, $\varepsilon \in \{1, -1\}^k$ and $Q_0 \in \operatorname{ind} \mathcal{A}$ such that for any simple S there exists an ε-path from Q_0 to S, then

$$\operatorname{gl.dim} \mathcal{A} \leq k+1 \text{ and } \operatorname{pd}_{\mathcal{A}} Q + \operatorname{id}_{\mathcal{A}} Q \leq k+2 \text{ for any } Q \in \operatorname{ind} \mathcal{A}$$

Corollary. If A is a finite dimensional, piecewise hereditary, *sincere* algebra, then

$$\operatorname{gl.dim} A \leq 3 \text{ and } \operatorname{pd}_A Q + \operatorname{id}_A Q \leq 4 \text{ for any indecomposable } Q$$

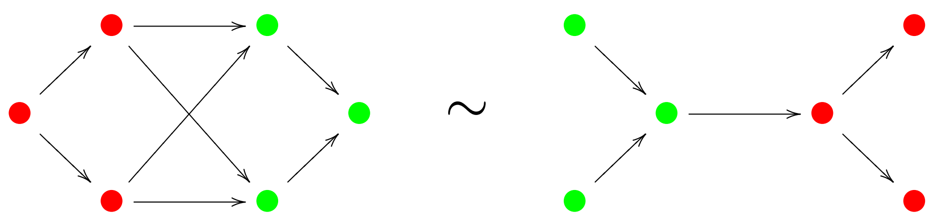
(M – sincere module, P_S – projective cover of S, use $M \leftarrow P_S \twoheadrightarrow S$).

Consequences for piecewise hereditary posets

Corollary. Let X be a poset. If kX is *piecewise hereditary*, then

$\text{gl.dim}\, kX \leq 3$ and $\text{pd}_{kX}\, Q + \text{id}_{kX}\, Q \leq 4$ for any indecomposable Q

Example 1. X with kX piecewise hereditary and $\text{gl.dim}\, kX = 3$.

$X =$ 

Example 2. $D_4 \times D_4$ is not piecewise hereditary:

$D_4 \times D_4 \sim A_2 \times A_2 \times A_2 \times A_2$ (use [Auslander-Platzeck-Reiten, Rickard])

The Euler form and Coxeter transformation

Λ – finite dimensional algebra, $\mathrm{gl.dim}\,\Lambda < \infty$

$\langle \cdot, \cdot \rangle_\Lambda : K_0(\Lambda) \times K_0(\Lambda) \to \mathbb{Z}$ – the *Euler form* of Λ

$$\langle M, N \rangle_\Lambda = \sum_{i \geq 0} (-1)^i \dim_k \mathrm{Ext}^i_\Lambda(M, N)$$

$\langle \cdot, \cdot \rangle_\Lambda$ is *positive* if $\langle x, x \rangle_\Lambda > 0$ for $x \neq 0$, *non-negative* if $\langle x, x \rangle_\Lambda \geq 0$.

Φ_Λ – the *Coxeter transformation* of Λ

$$\langle x, y \rangle_\Lambda = - \langle y, \Phi_\Lambda(x) \rangle_\Lambda$$

Φ_Λ is *periodic* if $\Phi_\Lambda^N = I$ for some N.

Periodicity of Φ_Λ and non-negativity of $\langle\cdot,\cdot\rangle_\Lambda$

Always: $\langle\cdot,\cdot\rangle_\Lambda$ positive $\Rightarrow$ Φ_Λ periodic.

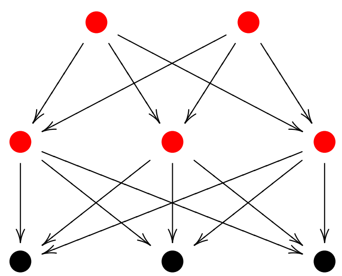
What about a converse?

Theorem. Λ – finite dimensional, *piecewise hereditary* algebra. Then

$$\Phi_\Lambda \text{ periodic} \Rightarrow \langle\cdot,\cdot\rangle_\Lambda \text{ non-negative}$$

Proof uses Happel's classification of hereditary categories with tilting object and [A'Campo, Ringel, Lenzing - de la Peña]

Example. A poset X with $\Phi_{kX}^6 = I$ but $\langle\cdot,\cdot\rangle_{kX}$ indefinite.



Weight types of canonical algebras

k – algebraically closed field

$\mathbf{p} = (p_1, \dots, p_t)$ – *weights*, $\boldsymbol{\lambda} = (\lambda_3, \dots, \lambda_t)$ – pairwise distinct in $k \setminus \{0\}$.

$\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$ – the *canonical algebra* of type $(\mathbf{p}, \boldsymbol{\lambda})$ [Ringel]

Theorem. Λ is derived equivalent to an *incidence algebra* of a poset if and only if $t \leq 3$ and $\mathbf{p} \neq (1, p)$.

Example. A poset whose incidence algebra is derived equivalent to the canonical algebra of type (p_1, p_2, p_3) with $p_i \geq 3$.