Perverse Morita equivalences are everywhere

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Perspective

Triangulated and *derived categories* can relate objects of different nature:

- *Coherent sheaves* over algebraic varieties and *modules* over non-commutative algebras [Beilinson 1978, Kapranov 1988]
- *Homological mirror symmetry conjecture* [Kontsevich 1994]

. . . but also relate non-isomorphic objects of the same nature:

- Morita theory for derived categories of modules [Rickard 1989]
- Derived categories of coherent sheaves [Bondal-Orlov 2002]
- *Broué's conjecture* on blocks of group algebras [Broué 1990]

Derived equivalence

Theorem [Rickard 1989]. Let R, S be rings. Then

$$\mathcal{D}(\mathsf{Mod}\, R) \simeq \mathcal{D}(\mathsf{Mod}\, S) \qquad (R, S \text{ are } \textit{derived equivalent},\ R \sim S)$$

if and only if there exists a *tilting complex* $T \in \mathcal{D}(\mathsf{Mod}\, R)$:

- *exceptional*: $\mathsf{Hom}_{\mathcal{D}(\mathsf{Mod}\, R)}(T, T[i]) = 0$ for $i \neq 0$,
- *compact generator*: $\langle \mathsf{add}\, T \rangle = \mathsf{per}\, R$,

such that $S \simeq \mathsf{End}_{\mathcal{D}(\mathsf{Mod}\, R)}(T)$.

Problems.

- No *decision* process.
- No *constructive* method.

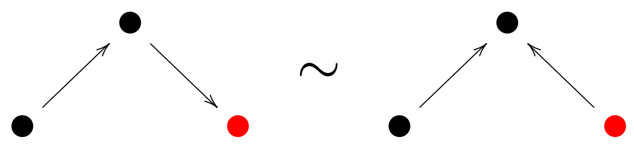
Example – BGP Reflections at sinks/sources

Q – quiver without oriented cycles,
s – *sink* in Q, i.e. no outgoing arrows from s.

$\sigma_s Q$ – the *BGP reflection* with respect to s, obtained from Q by inverting all arrows incident to s, so that s becomes a *source*.

Theorem [Bernstein-Gelfand-Ponomarev]. $KQ \sim K\sigma_s Q$.

Example.



Remark. Generalized by [Auslander-Platzeck-Reiten] to sinks in quivers of arbitrary finite-dimensional algebras.

Outline

BGP reflection is a *combinatorial*, *local* operation at *sinks/sources* *producing derived equivalences* for *path algebras* of quivers.

We will present generalizations for

- Arbitrary finite-dimensional algebras,
- at arbitrary vertices;

and explore their connections with

- *Perverse Morita equivalences* [Chuang-Rouquier],
- *Quiver mutation* [Fomin-Zelevinsky 2002].

From vertices to complexes

K – algebraically closed field,
$A = KQ/I$ – quiver with relations,

vertex $i \rightsquigarrow$ projective P_i, arrow $i \to j \rightsquigarrow$ map $P_j \to P_i$

k – vertex in Q without loops,

$$T_k^- = \Big(P_k \to \bigoplus_{j \to k} P_j\Big) \oplus \bigoplus_{i \neq k} P_i, \qquad T_k^+ = \Big(\bigoplus_{k \to j} P_j \to P_k\Big) \oplus \bigoplus_{i \neq k} P_i$$

Are these *tilting complexes?*

- Always *compact generators*,
- *Exceptionality* is expressed in terms of the combinatorial data.

Mutations of algebras

If T_k^- is a tilting complex, the *negative mutation* at k is defined as

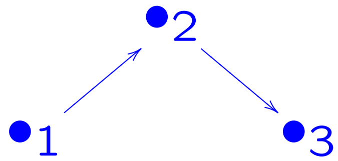
$$\mu_k^-(A) = \mathrm{End}_{\mathcal{D}(A)}(T_k^-)$$

If T_k^+ is a tilting complex, the *positive mutation* at k is defined as

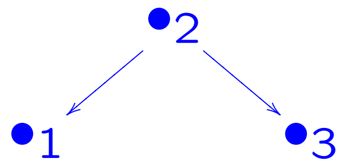
$$\mu_k^+(A) = \mathrm{End}_{\mathcal{D}(A)}(T_k^+)$$

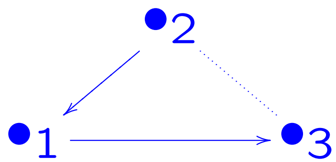
- There are *up to two* mutations at a vertex,
- Mutations yield *derived equivalent* algebras,
- Closely related to the *Brenner-Butler* tilting modules,
- Mutations are *perverse Morita equivalences* [Chuang-Rouquier].

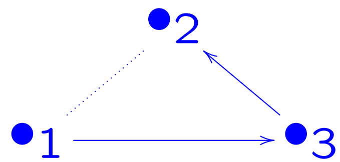
Mutations of algebras – Example

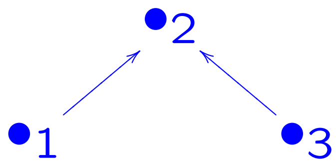
$A =$ 

$\mu_1^-(A)$ is not defined

$\mu_1^+(A) =$ 

$\mu_2^-(A) =$ 

$\mu_2^+(A) =$ 

$\mu_3^-(A) =$ 

$\mu_3^+(A)$ is not defined

Remark. For $A' = \mu_2^-(A)$, neither $\mu_1^-(A')$ nor $\mu_1^+(A')$ are defined.

Brenner-Butler tilting and algebra mutations

$$T_k^{\mathsf{BB}} = \tau_A^- S_k \oplus \bigoplus_{i \neq k} P_i$$

If T_k^{BB} is a tilting module, the *BB-mutation* at k is defined as

$$\mu_k^{\mathsf{BB}}(A) = \mathsf{End}_A(T_k^{\mathsf{BB}})$$

- BB-mutation *is* the negative mutation,
- Under mild conditions, the converse holds.

n-*BB-tilting* [Hu-Xi 2008] and n-*APR-tilting* [Iyama-Oppermann 2009] can be written as composition of n negative mutations at the same vertex

$$A' \underset{n-\mathsf{BB}}{\sim} A \quad \Rightarrow \quad A' \simeq \mu_k^- \mu_k^- \dots \mu_k^-(A)$$

Perverse Morita equivalences [Chuang-Rouquier]

$F : \mathcal{D}^b(A) \xrightarrow{\sim} \mathcal{D}^b(A')$ is a *perverse Morita equivalence* if there are:

- Filtrations of the simples $\phi \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_r$, $\phi \subset \mathcal{S}'_1 \subset \cdots \subset \mathcal{S}'_r$,
- *Perversity function* $p : \{1, \ldots, r\} \to \mathbb{Z}$,

such that F induces equivalences

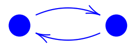
$$F : \mathcal{D}^b_{\mathcal{A}_i}(A) \simeq \mathcal{D}^b_{\mathcal{A}'_i}(A') \qquad F[-p(i)] : \mathcal{A}_i/\mathcal{A}_{i-1} \simeq \mathcal{A}'_i/\mathcal{A}'_{i-1}$$

where $\mathcal{A}_i, \mathcal{A}'_i$ are the Serre subcategories generated by $\mathcal{S}_i, \mathcal{S}'_i$.

When $\mu_k^-(A)$ is defined, then $\mathbf{R}\mathrm{Hom}_A(T_k^-, -) : \mathcal{D}^b(A) \xrightarrow{\sim} \mathcal{D}^b(\mu_k^-(A))$ is a perverse Morita equivalence, with

- Filtration $\phi \subset \{k\} \subset \{1, 2, \ldots, n\}$,
- Perversity $\{-1, 0\}$.

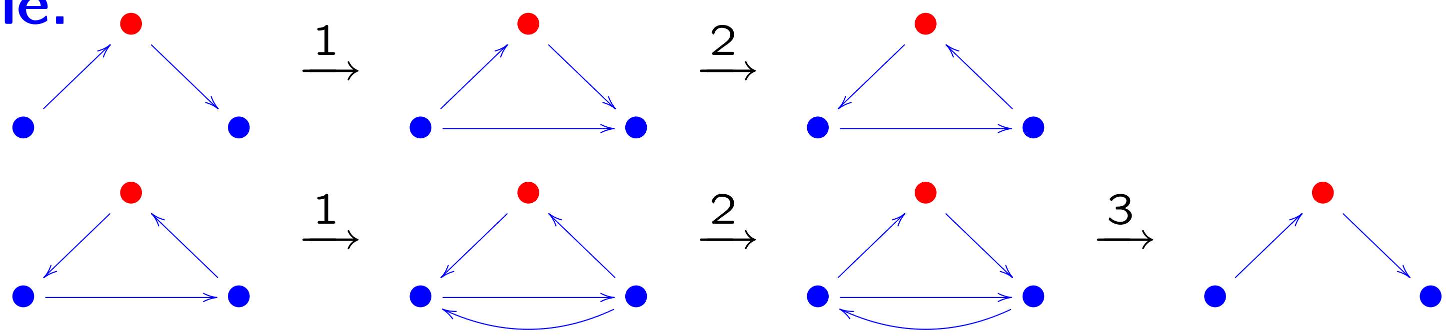
Quiver mutation [Fomin-Zelevinsky]

Q – quiver without *loops* () and *2-cycles* (),
k – any vertex in Q.

The *mutation* of Q at k, denoted $\mu_k(Q)$, is obtained as follows:

1. For any pair $i \xrightarrow{\alpha} k \xrightarrow{\beta} j$, add new arrow $i \xrightarrow{[\alpha\beta]} j$,
2. Invert the incoming and outgoing arrows at k,
3. Remove a maximal set of 2-cycles.

Example.



Quiver mutation – matrix version

$Q \rightsquigarrow B_Q$, via $(B_Q)_{ij} = |\{\text{arrows } j \to i\}| - |\{\text{arrows } i \to j\}|$, induces

$\{$*quivers*, no loops and 2-cycles$\} \leftrightarrow \{$*anti-symmetric* integral *matrices*$\}$

Lemma [FZ, Geiss-Leclerc-Schröer]. If Q has no loops and 2-cycles,

$$(r_k^-)^T \cdot B_Q \cdot r_k^- = B_{\mu_k(Q)} = (r_k^+)^T \cdot B_Q \cdot r_k^+$$

where

$$r_k^- = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ * & * & -1 & * & * \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad r_k^+ = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ * & * & -1 & * & * \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$(r_k^-)_{kj} = |\{\text{arrows } j \to k\}|, \quad (r_k^+)_{kj} = |\{\text{arrows } k \to j\}|, \quad (j \neq k).$$

Mutations, Cartan matrices and Euler forms

C_A – the *Cartan matrix* of A, defined by $(C_A)_{ij} = \dim_K \operatorname{Hom}_A(P_i, P_j)$.

The *bilinear form* defined by C_A is invariant under derived equivalence.

Lemma.

$$C_{\mu_k^-(A)} = r_k^- \cdot C_A \cdot (r_k^-)^T \qquad C_{\mu_k^+(A)} = r_k^+ \cdot C_A \cdot (r_k^+)^T$$

whenever the mutations are defined.

When A has *finite global dimension*, its *Euler form* is $c_A = C_A^{-T}$, and

$$c_{\mu_k^-(A)} = (r_k^-)^T \cdot c_A \cdot r_k^- \qquad c_{\mu_k^+(A)} = (r_k^+)^T \cdot c_A \cdot r_k^+$$

whenever the mutations are defined.

Mutations of algebras as mutations of quivers

When the discrete data associated to an algebra – *quiver* and *Euler form*, are "compatible", mutation of algebras *is* mutation of quivers.

This happens for:

- Algebras of *global dimension 2*,
- Endomorphism algebras of cluster-tilting objects in *stably 2-CY Frobenius* categories [Buan-Iyama-Reiten-Scott 2009, GLS, Palu 2009].

Mutations of algebras of global dimension 2

A – finite-dimensional K-algebra of *global dimension* ≤ 2.

The *extended quiver* $\tilde{Q}_A$ [Assem-Brüstle-Schiffler 2008, Keller] has

$$|\{\text{arrows } i \to j\}| = \dim_K \operatorname{Ext}^1_A(S_i, S_j) + \dim_K \operatorname{Ext}^2_A(S_j, S_i)$$

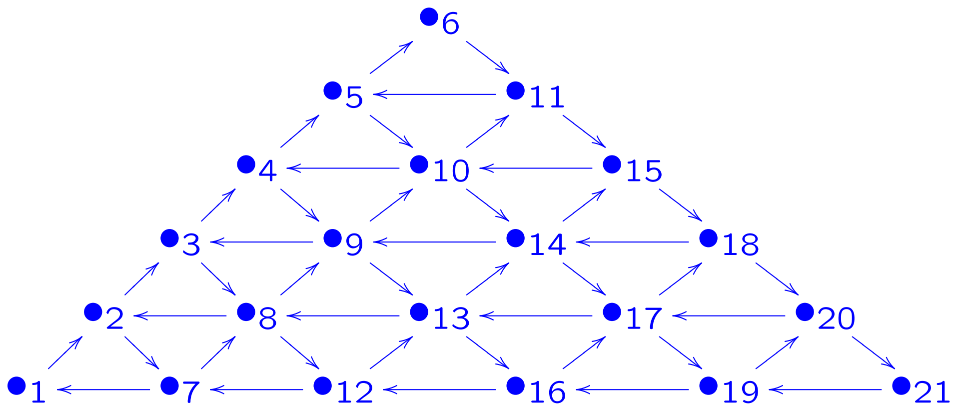
so that $B_{\tilde{Q}_A} = c_A - c_A^T$ is the *anti-symmetrization* of c_A.

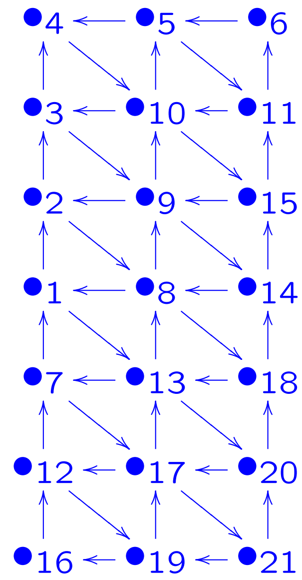
Theorem [L]. Assume that $\tilde{Q}_A$ is without loops and 2-cycles.
If $\mu_k^-(A)$ is defined and $\operatorname{gl.dim} \mu_k^-(A) \leq 2$, then $\tilde{Q}_{\mu_k^-(A)} = \mu_k(\tilde{Q}_A)$.
If $\mu_k^+(A)$ is defined and $\operatorname{gl.dim} \mu_k^+(A) \leq 2$, then $\tilde{Q}_{\mu_k^+(A)} = \mu_k(\tilde{Q}_A)$.

Remark. Not all quiver mutations correspond to algebra mutations.

Example – Sequence of mutations

Theorem [L]. $\widetilde{Q}_{\mathrm{Aus}(\overrightarrow{A_{2n}})}$ and $\widetilde{Q}_{K(A_n\times A_{2n+1})}$ are *mutation equivalent*.





$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 1, 2, 3, 4, 5, 12, 13, 14, 15, 7, 8, 9, 10, 1, 2, 3, 4,$
$21, 19, 16, 20, 17, 12, 18, 13, 7, 21, 19, 16, 20, 17, 12, 21, 19, 16$

Mutations in (stably) 2-Calabi-Yau categories

$\mathcal{C}$ – Hom-finite, *triangulated 2-CY* or *Frobenius stably 2-CY*,

U – *cluster-tilting object* in $\mathcal{C}$ [Buan-Marsh-Reineke-Reiten-Todorov, GLS]

Three notions of mutation:

$U \rightsquigarrow \mu_k(U)$	mutation of *CT objects*,
$\Lambda = \mathrm{End}_{\mathcal{C}}(U) \rightsquigarrow \Lambda' = \mathrm{End}_{\mathcal{C}}(\mu_k(U))$	mutation of *algebras*?
$Q_\Lambda \rightsquigarrow Q_{\Lambda'}$	mutation of *quivers*.

studied in [BIRSc, BIRSm, BMRRT 2006, GLS, Iyama-Yoshino 2008].

Mutations in Frobenius stably 2-CY categories

In the *Frobenius* case, mutation of cluster-tilting objects leads to mutation of their endomorphism algebras:

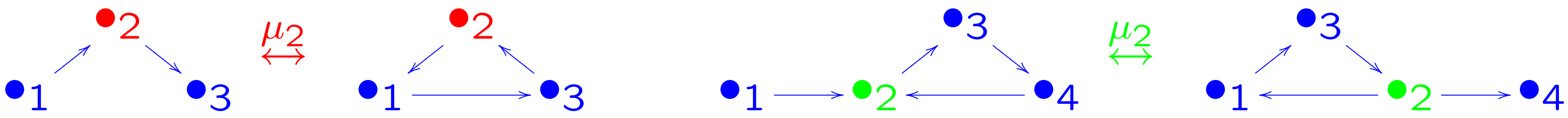
Theorem [L]. If Q_Λ and $Q_{\Lambda'}$ have no loops at k, then

$$\Lambda' \simeq \mu_k^{\mathrm{BB}}(\Lambda) \simeq \mu_k^{-}(\Lambda) \simeq \mu_k^{+}(\Lambda)$$

- Generalizes [GLS],
- Builds on [Hu-Xi 2008],
- Reminiscent of [Iyama-Reiten 2008] for 3-CY algebras,
- The matrix c_Λ is "partially" skew-symmetric [Keller-Reiten 2007].

Mutations for 2-CY-tilted algebras

In the *triangulated* case, mutation of cluster-tilting objects does not always lead to mutation of their endomorphism algebras:

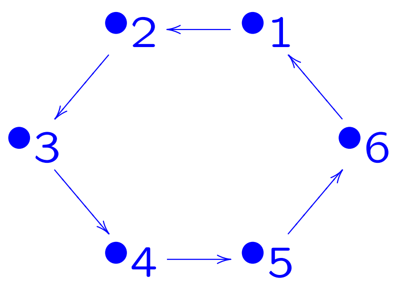


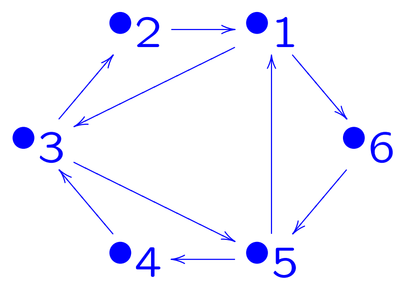
Theorem [L]. $\Lambda' \simeq \mu_k^{\mathrm{BB}}(\Lambda) \iff \mu_k^{\mathrm{BB}}(\Lambda)$ and $\mu_k^{\mathrm{BB}}(\Lambda'^{op})$ are defined.

- There is an effective *algorithm* that decides whether $\Lambda' \simeq \mu_k^{\mathrm{BB}}(\Lambda)$.
- Has been applied in [Bastian-Holm-L.] for the derived equivalence classification of *cluster-tilted algebras* [BMR 2007] of various classes.
- Applicable also for finite-dimensional *Jacobian* algebras [Amiot 2009, Derksen-Weyman-Zelevinsky 2008].

More perverse Morita equivalences

- *n-BB-tilting*: $\phi \subset \{k\} \subset \mathcal{S}$, perversity $\{-n, 0\}$.
- *Hughes-Waschbüsch reflection*: $\phi \subset \mathcal{S} \setminus \{k\} \subset \mathcal{S}$, perversity $\{-1, 0\}$.
- **Example.** $\phi \subset \{2, 4, 6\} \subset \{1, 2, 3, 4, 5, 6\}$, perversity $\{-1, 0\}$,





Generally, for *any* $\phi \subset I \subset \mathcal{S}$ and *any* perversity, one can construct a complex such that if it is tilting, it induces perverse equivalence.

Question. Can any two *derived equivalent* algebras be connected by a sequence of *perverse Morita equivalences*?