Graduate Seminar on Advanced Geometry S4D3
Summer semester 2021

Topic: Kazhdan's property T

The seminar builds on the material covered in the class on Geometric Group Theory by Elia Fioravanti in WS20/21

Further requirements: Some basic knowledge on geometry, topology and measure theory

Summary: Kazhdan formulated a property for representations of (topological) groups which turned out to be of far-reaching insight. This property is called property (T) and can be defined as follows.

Let G be a topological group, and let $\rho : G \to U(\mathcal{H})$ be a strongly continuous homomorphism into the unitary group $U(\mathcal{H})$ of a Hilbert space $\mathcal{H}$. The representation ρ has an *almost invariant vector* if for any compact subset $Q \subset G$ and any $\epsilon > 0$, there exists a vector $0 \neq v \in \mathcal{H}$ such that for all $g \in Q$, we have

$$\|v - \rho(g)v\| \leq \epsilon \|v\|.$$

The group G has property (T) if the existence of an almost invariant vector for ρ implies the existence of an invariant vector, that is, a vector $v \neq 0$ with $\rho(g)v = v$ for all $g \in G$.

While this definition seems abstract and purely representation theoretic, it has a large number of consequences for geometric actions and purely group theoretical as well. For example, if G is locally compact (e.g. discrete) and has property (T), then G is compactly generated (e.g. finitely generated).

Examples of groups with property (T) are the simple Lie groups of non-compact type and rank at least two as well as the quaternionic hyperbolic groups. The property is inherited by lattices and quotient groups. It enters in many results on geometric group actions. The most recent spectacular addition to the collection of groups with this property is the automorphism group of the free group on at least 5 generators.

The goal of the seminar is to obtain some understanding of property (T) from the eyes of a geometer.

A preliminary meeting will be held via Zoom on Feb. 8 at 12.00h. The Zoom information is available on Basis.

Talk 1: Introduction of Property (T). Characterization of property (T) in terms of the Fell topology for unitary representations.
[BHV08] pages 27–33, 403-404.

Talk 2: First applications: Compact generation; property (T) for pairs.
[BHV08] pages 36-43.

Talk 3: Property (T) for $SL(n, \mathbb{Z})$, $n \geq 3$.
[BHV08] pages 44-50.

Talk 4: Algebraic consequences of property (T): Vanishing first cohomology, inheritance for lattices.
[BHV08] pages 60-67.

Talk 5: First cohomology, affine actions and property (FH)
[BHV08] pages 74-81, 85-88.

Talk 6: Actions of groups on trees; property (FA)
[BHV08] pages 81-85, [S03] pages 58-61.

Talk 7: Functions conditionally of negative type; discrete groups of isometries of real hyperbolic space
[BHV08] pages 119-127.

Talk 8: A spectral criterion for property (T)
[BHV08] pages 236-245.

References

[BHV08] B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's property (T)*, new mathematical monographs, Cambridge Univ. Press, Cambridge 2008.
[S03] J. P. Serre, *Trees*, Springer Monographs in Math. 2003.